34. MATHEMATICAL TOOLS

***** This contains additions and sections by Dr. Andrew Sterian.

• We use math in almost every problem we solve. As a result the more relevant topics of mathematics are summarized here.

• This is not intended for learning, but for reference.

34.1 INTRODUCTION

• This section has been greatly enhanced, and tailored to meet our engineering requirements.

• The section outlined here is not intended to teach the elements of mathematics, but it is designed to be a quick reference guide to support the engineer required to use techniques that may not have been used recently.

• For those planning to write the first ABET Fundamentals of Engineering exam, the following topics are commonly on the exam.

- quadratic equation
- straight line equations - slop and perpendicular
- conics, circles, ellipses, etc.
- matrices, determinants, adjoint, inverse, cofactors, multiplication
- limits, L'Hospital's rule, small angle approximation
- integration of areas
- complex numbers, polar form, conjugate, addition of polar forms
- maxima, minima and inflection points
- first-order differential equations - guessing and separation
- second-order differential equation - linear, homogeneous, non-homogeneous, second-order
- triangles, sine, cosine, etc.
- integration - by parts and separation
- solving equations using inverse matrices, Cramer's rule, substitution
- eigenvalues, eigenvectors
- dot and cross products, areas of parallelograms, angles and triple product
- divergence and curl - solenoidal and conservative fields
- centroids
- integration of volumes

- integration using Laplace transforms
- probability - permutations and combinations
- mean, standard deviation, mode, etc.
- log properties
- taylor series
- partial fractions
- basic coordinate transformations - cartesian, cylindrical, spherical
- trig identities
- derivative - basics, natural log, small angles approx., chain rule, partial fractions

34.1.1 Constants and Other Stuff

• A good place to start a short list of mathematical relationships is with greek letters

lower case	upper case	name
α	Α	alpha
β	Β	beta
γ	Γ	gamma
δ	Δ	delta
ε	Ε	epsilon
ζ	Ζ	zeta
η	Η	eta
θ	Θ	theta
ι	Ι	iota
κ	Κ	kappa
λ	Λ	lambda
μ	Μ	mu
ν	Ν	nu
ξ	Ξ	xi
ο	Ο	omicron
π	Π	pi
ρ	Ρ	rho
σ	Σ	sigma
τ	Τ	tau
υ	Υ	upsilon
φ	Φ	phi
χ	Χ	chi
ψ	Ψ	psi
ω	Ω	omega

Figure 34.1 The greek alphabet

• The constants listed are amount some of the main ones, other values can be derived through calculation using modern calculators or computers. The values are typically given with more than 15 places of accuracy so that they can be used for double precision calculations.

$$e = 2.7182818 = \lim_{n \to \infty}\left(1 + \frac{1}{n}\right)^n = \text{natural logarithm base}$$

$$\pi = 3.1415927 = \text{pi}$$

$$\gamma = 0.57721566 = \text{Eulers constant}$$

$$1 radian = 57.29578°$$

Figure 34.2 Some universal constants

34.1.2 Basic Operations

• These operations are generally universal, and are described in sufficient detail for our use.

• Basic properties include,

commutative	$a + b = b + a$	
distributive	$a(b + c) = ab + ac$	
associative	$a(bc) = (ab)c$	$a + (b + c) = (a + b) + c$

Figure 34.3 Basic algebra properties

34.1.2.1 - Factorial

• A compact representation of a series of increasing multiples.

$$n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \ldots \cdot n$$

$$0! = 1$$

Figure 34.4 The basic factorial operator

34.1.3 Exponents and Logarithms

• The basic properties of exponents are so important they demand some sort of mention

$$(x^n)(x^m) = x^{n+m} \qquad x^0 = 1 \text{ , if x is not 0} \qquad x^{\frac{1}{n}} = \sqrt[n]{x}$$

$$\frac{(x^n)}{(x^m)} = x^{n-m} \qquad x^{-p} = \frac{1}{x^p} \qquad x^{\frac{m}{n}} = \sqrt[n]{x^m}$$

$$(x^n)^m = x^{n \cdot m} \qquad (xy)^n = (x^n)(y^n) \qquad \sqrt[n]{\frac{x}{y}} = \frac{\sqrt[n]{x}}{\sqrt[n]{y}}$$

Figure 34.5 Properties of exponents

• Logarithms also have a few basic properties of use, The basic base 10 logarithm:

$$\log x = y \qquad x = 10^y$$

The basic base n logarithm:

$$\log_n x = y \qquad x = n^y$$

The basic natural logarithm (e is a constant with a value found near the start of this section:

$$\ln x = \log_e x = y \qquad x = e^y$$

Figure 34.6 Definitions of logarithms

• All logarithms observe a basic set of rules for their application,

$$\log_n(xy) = \log_n(x) + \log_n(y) \qquad \log_n(n) = 1$$

$$\log_n(1) = 0$$

$$\log_n\left(\frac{x}{y}\right) = \log_n(x) - \log_n(y)$$

$$\log_n(x^y) = y\log_n(x)$$

$$\log_n(x) = \frac{\log_m(x)}{\log_m(n)}$$

$$\ln(A\angle\theta) = \ln(A) + (\theta + 2\pi k)j \qquad k \in I$$

Figure 34.7 Properties of logarithms

34.1.4 Polynomial Expansions

• Binomial expansion for polynomials,

$$(a + x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{2!}a^{n-2}x^2 + \ldots + x^n$$

Figure 34.8 A general expansion of a polynomial

34.1.5 Practice Problems

1. Are the following expressions equivalent?

a) $A(5 + B) - C = 5A + B - C$

b) $\frac{A+B}{C+D} = \frac{A}{C} + \frac{B}{D}$

c) $\log(ab) = \log(a) + \log(b)$

d) $5(5^4) = 5^5$

e) $3\log(4) = \log(16)$

f) $(x + 6)(x - 6) = x^2 + 36$

g) $10^{\log(5)} = \frac{10}{5}$

h) $\sqrt{\frac{(x+1)^6}{(x+1)^2}} = x^2 + 2x + 1$

2. Simplify the following expressions.

a) $x(x + 2)^2 - 3x$

b) $\frac{(x+3)(x+1)x^2}{(x+1)^2 x}$

c) $\log(x^3)$

d) $\frac{64}{16}$

e) $\frac{15}{21} + \frac{3}{28}$

f) $(x^2 y^3)^4$

g) $\sqrt{4x^2 - 8y^4}$

h) $\frac{5}{3}\left(\frac{8}{9}\right)$

i) $\frac{\left(\frac{5}{4}\right)}{5}$

j) $(y + 4)^3(y - 2)$

k) $\sqrt{x^2 y}$

l) $\frac{x+1}{x+2} = 4$

3. Simplify the following expressions.

a) $\frac{A+B}{AB}$

b) $\frac{AB}{A+B}$

c) $\left(\frac{(x^4y^5)}{x^2}\right)^3$

d) $\log(x^5) + \log(x^3)$

(ans.

a) $\frac{A+B}{AB} = \frac{A}{AB} + \frac{B}{AB} = \frac{1}{B} + \frac{1}{A}$

b) $\frac{AB}{A+B}$ cannot be simplified

c) $\left(\frac{(x^4y^5)}{x^2}\right)^3 = (x^2y^5)^3 = x^6y^{15}$

d) $\log(x^5) + \log(x^3) = 5\log(x) + 3\log(x) = 8\log(x)$

4. Simplify the following expressions.

a) $n\log(x^2) + m\log(x^3) - \log(x^4)$

(ans.

a) $n\log(x^2) + m\log(x^3) - \log(x^4)$

$2n\log(x) + 3m\log(x) - 4\log(x)$

$(2n + 3m - 4)\log(x)$

$(2n + 3m - 4)\log(x)$

5. Rearrange the following equation so that only 'y' is on the left hand side.

$$\frac{y+x}{y+z} = x+2$$

(ans.

$$\frac{y+x}{y+z} = x+2$$

$$y+x = (x+2)(y+z)$$

$$y+x = xy+xz+2y+2z$$

$$y-xy-2y = xz+2z-x$$

$$y(-x-1) = xz+2z-x$$

$$y = \frac{xz+2z-x}{-x-1}$$

6. Find the limits below.

a) $$\lim_{t\to 0}\left(\frac{t^3+5}{5t^3+1}\right)$$

b) $$\lim_{t\to \infty}\left(\frac{t^3+5}{5t^3+1}\right)$$

(ans.

a) $$\lim_{t\to 0}\left(\frac{t^3+5}{5t^3+1}\right) = \frac{0^3+5}{5(0)^3+1} = 5$$

b) $$\lim_{t\to \infty}\left(\frac{t^3+5}{5t^3+1}\right) = \frac{\infty^3+5}{5(\infty)^3+1} = \frac{\infty^3}{5(\infty)^3} = 0.2$$

34.2 FUNCTIONS

34.2.1 Discrete and Continuous Probability Distributions

Binomial

$$P(m) = \sum_{t \le m} \binom{n}{t} p^t q^{n-t} \qquad q = 1 - p \qquad q, p \in [0, 1]$$

Poisson

$$P(m) = \sum_{t \le m} \frac{\lambda^t e^{-\lambda}}{t!} \qquad \lambda > 0$$

Hypergeometric

$$P(m) = \sum_{t \le m} \frac{\binom{r}{t}\binom{s}{n-t}}{\binom{r+s}{n}}$$

Normal

$$P(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2} dt$$

Figure 34.9 Distribution functions

34.2.2 Basic Polynomials

• The quadratic equation appears in almost every engineering discipline, therefore is of great importance.

$$ax^2 + bx + c = 0 = a(x - r_1)(x - r_2) \qquad r_1, r_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Figure 34.10 Quadratic equation

• Cubic equations also appear on a regular basic, and as a result should also be considered.

$$x^3 + ax^2 + bx + c = 0 = (x - r_1)(x - r_2)(x - r_3)$$

First, calculate,

$$Q = \frac{3b - a^2}{9} \qquad R = \frac{9ab - 27c - 2a^3}{54} \qquad S = \sqrt[3]{R + \sqrt{Q^3 + R^2}} \qquad T = \sqrt[3]{R - \sqrt{Q^3 + R^2}}$$

Then the roots,

$$r_1 = S + T - \frac{a}{3} \qquad r_2 = \frac{S+T}{2} - \frac{a}{3} + \frac{j\sqrt{3}}{2}(S - T) \qquad r_3 = \frac{S+T}{2} - \frac{a}{3} - \frac{j\sqrt{3}}{2}(S - T)$$

Figure 34.11 Cubic equations

• On a few occasions a quartic equation will also have to be solved. This can be done by first reducing the equation to a quadratic,

$$x^4 + ax^3 + bx^2 + cx + d = 0 = (x - r_1)(x - r_2)(x - r_3)(x - r_4)$$

First, solve the equation below to get a real root (call it 'y'),

$$y^3 - by^2 + (ac - 4d)y + (4bd - c^2 - a^2 d) = 0$$

Next, find the roots of the 2 equations below,

$$r_1, r_2 = z^2 + \left(\frac{a + \sqrt{a^2 - 4b + 4y}}{2}\right)z + \left(\frac{y + \sqrt{y^2 - 4d}}{2}\right) = 0$$

$$r_3, r_4 = z^2 + \left(\frac{a - \sqrt{a^2 - 4b + 4y}}{2}\right)z + \left(\frac{y - \sqrt{y^2 - 4d}}{2}\right) = 0$$

Figure 34.12 Quartic equations

34.2.3 Partial Fractions

• The next is a flowchart for partial fraction expansions.

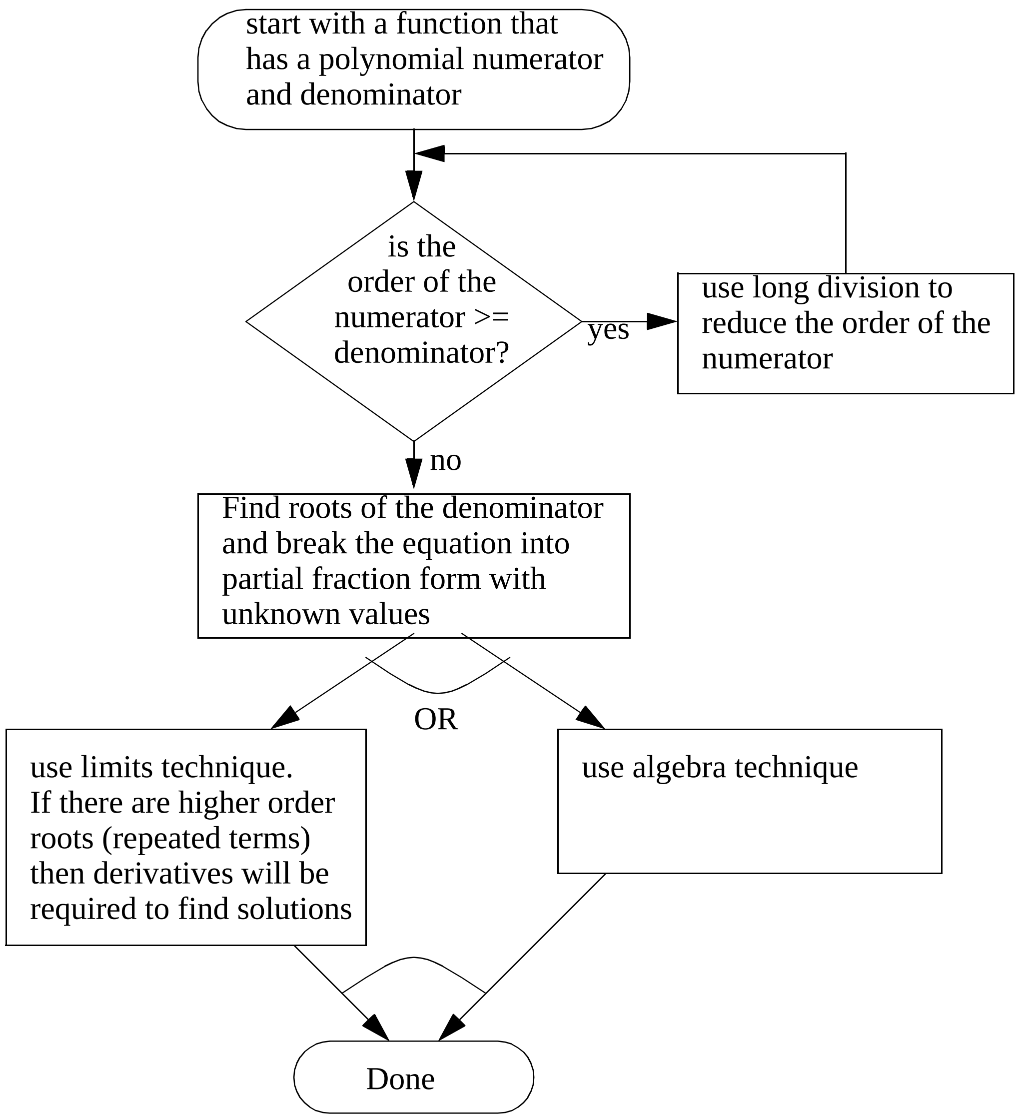


Figure 34.13 The methodolgy for solving partial fractions

• The partial fraction expansion for,

$$x(s) = \frac{1}{s^2(s+1)} = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{s+1}$$

$$C = \lim_{s \to -1}\left[(s+1)\left(\frac{1}{s^2(s+1)}\right)\right] = 1$$

$$A = \lim_{s \to 0}\left[s^2\left(\frac{1}{s^2(s+1)}\right)\right] = \lim_{s \to 0}\left[\frac{1}{s+1}\right] = 1$$

$$B = \lim_{s \to 0}\left[\frac{d}{ds}\left[s^2\left(\frac{1}{s^2(s+1)}\right)\right]\right] = \lim_{s \to 0}\left[\frac{d}{ds}\left(\frac{1}{s+1}\right)\right] = \lim_{s \to 0}[-(s+1)^{-2}] = -1$$

Figure 34.14 A partial fraction example

• Consider the example below where the order of the numerator is larger than the denominator.

$$x(s) = \frac{5s^3 + 3s^2 + 8s + 6}{s^2 + 4}$$

This cannot be solved using partial fractions because the numerator is 3rd order and the denominator is only 2nd order. Therefore long division can be used to reduce the order of the equation.

$$\begin{array}{r|l} & 5s + 3 \\ \hline s^2 + 4 & 5s^3 + 3s^2 + 8s + 6 \\ & 5s^3 + 20s \\ \hline & 3s^2 - 12s + 6 \\ & 3s^2 + 12 \\ \hline & -12s - 6 \end{array}$$

This can now be used to write a new function that has a reduced portion that can be solved with partial fractions.

$$x(s) = 5s + 3 + \frac{-12s - 6}{s^2 + 4} \qquad \text{solve} \qquad \frac{-12s - 6}{s^2 + 4} = \frac{A}{s + 2j} + \frac{B}{s - 2j}$$

Figure 34.15 Solving partial fractions when the numerator order is greater than the denominator

• When the order of the denominator terms is greater than 1 it requires an expanded partial fraction form, as shown below.

$$F(s) = \frac{5}{s^2(s+1)^3}$$

$$\frac{5}{s^2(s+1)^3} = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{(s+1)^3} + \frac{D}{(s+1)^2} + \frac{E}{(s+1)}$$

Figure 34.16 Partial fractions with repeated roots

• We can solve the previous problem using the algebra technique.

$$\frac{5}{s^2(s+1)^3} = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{(s+1)^3} + \frac{D}{(s+1)^2} + \frac{E}{(s+1)}$$

$$= \frac{A(s+1)^3 + Bs(s+1)^3 + Cs^2 + Ds^2(s+1) + Es^2(s+1)^2}{s^2(s+1)^3}$$

$$= \frac{s^4(B+E) + s^3(A+3B+D+2E) + s^2(3A+3B+C+D+E) + s(3A+B) + (A)}{s^2(s+1)^3}$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 3 & 0 & 1 & 2 \\ 3 & 3 & 1 & 1 & 1 \\ 3 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 5 \end{bmatrix} \qquad \begin{bmatrix} A \\ B \\ C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 3 & 0 & 1 & 2 \\ 3 & 3 & 1 & 1 & 1 \\ 3 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 5 \\ -15 \\ 5 \\ 10 \\ 15 \end{bmatrix}$$

$$\frac{5}{s^2(s+1)^3} = \frac{5}{s^2} + \frac{-15}{s} + \frac{5}{(s+1)^3} + \frac{10}{(s+1)^2} + \frac{15}{(s+1)}$$

Figure 34.17 An algebra solution to partial fractions

34.2.4 Summation and Series

• The notation $\sum_{i=a}^{b} x_i$ is equivalent to $x_a + x_{a+1} + x_{a+2} + \ldots + x_b$ assuming a and b are integers and $b \geq a$. The index variable i is a placeholder whose name does not matter.

• Operations on summations:

$$\sum_{i=a}^{b} x_i = \sum_{i=b}^{a} x_i$$

$$\sum_{i=a}^{b} \alpha x_i = \alpha \sum_{i=a}^{b} x_i$$

$$\sum_{i=a}^{b} x_i + \sum_{j=a}^{b} y_j = \sum_{i=a}^{b} (x_i + y_i)$$

$$\sum_{i=a}^{b} x_i + \sum_{i=b+1}^{c} x_i = \sum_{i=a}^{c} x_i$$

$$\left(\sum_{i=a}^{b} x_i\right)\left(\sum_{j=c}^{d} y_j\right) = \sum_{i=a}^{b} \sum_{j=c}^{d} x_i y_j$$

• Some common summations:

$$\sum_{i=1}^{N} i = \frac{1}{2}N(N+1)$$

$$\sum_{i=0}^{N-1} \alpha^i = \begin{cases} \dfrac{1-\alpha^N}{1-\alpha}, & \alpha \neq 1 \\ N, & \alpha = 1 \end{cases}$$ for both real and complex α.

$$\sum_{i=0}^{\infty} \alpha^i = \frac{1}{1-\alpha}, \ |\alpha| < 1$$ for both real and complex α. For $|\alpha| \geq 1$, the summation does not converge.

34.2.5 Practice Problems

1. Convert the following polynomials to multiplied terms as shown in the example.

e.g., $x^2 + 2x + 1 = (x+1)(x+1)$

a) $x^2 - 2x + 1$

b) $x^2 - 1$

c) $x^2 + 1$

d) $x^2 + x + 10$

e)

f)

2. Solve the following equation to find ‘x’.

$$2x^2 + 8x = -8$$

(ans.

$$2x^2 + 8x = -8$$
$$x^2 + 4x + 4 = 0$$
$$(x+2)^2 = 0$$
$$x = -2, -2$$

3. Reduce the following expression to partial fraction form.

$$\frac{x+4}{x^2+8x}$$

(ans. $\frac{x+4}{x^2+8x} = \frac{x+4}{x(x+8)} = \frac{A}{x} + \frac{B}{x+8} = \frac{Ax + 8A + Bx}{x^2+8x}$

$$(1)x + 4 = (A+B)x + 8A$$

$8A = 4$ $\quad A = 0.5$

$1 = A + B$ $\quad B = 0.5$

$$\frac{x+4}{x^2+8x} = \frac{0.5}{x} + \frac{0.5}{x+8}$$

34.3 SPATIAL RELATIONSHIPS

34.3.1 Trigonometry

• The basic trigonometry functions are,

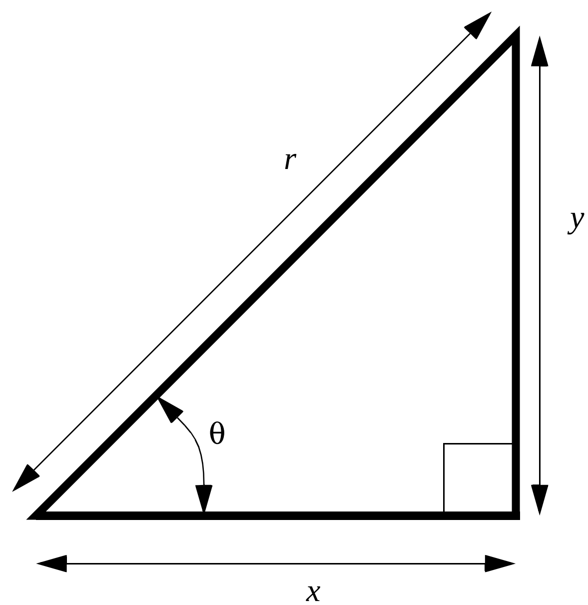
$$\sin\theta = \frac{y}{r} = \frac{1}{\csc\theta}$$

$$\cos\theta = \frac{x}{r} = \frac{1}{\sec\theta}$$

$$\tan\theta = \frac{y}{x} = \frac{1}{\cot\theta} = \frac{\sin\theta}{\cos\theta}$$

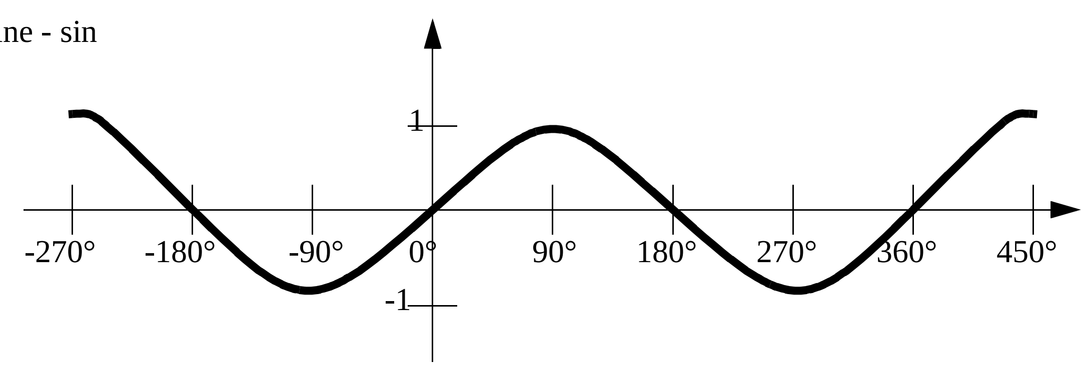
Pythagorean Formula:

$$r^2 = x^2 + y^2$$

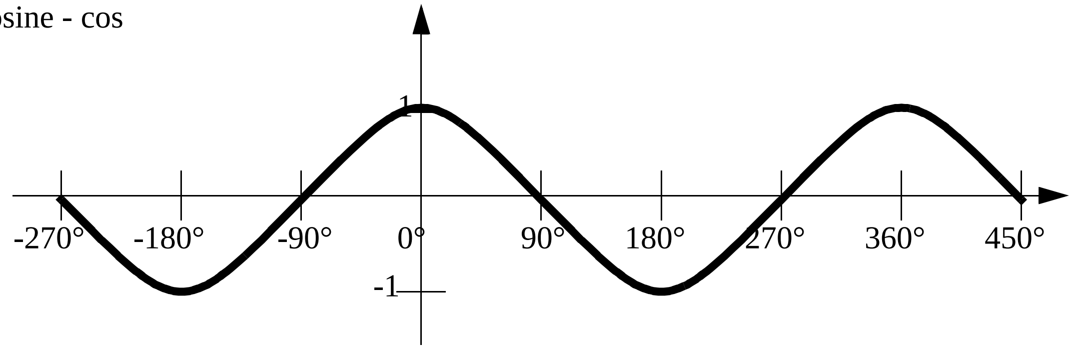


• Graphs of these functions are given below,

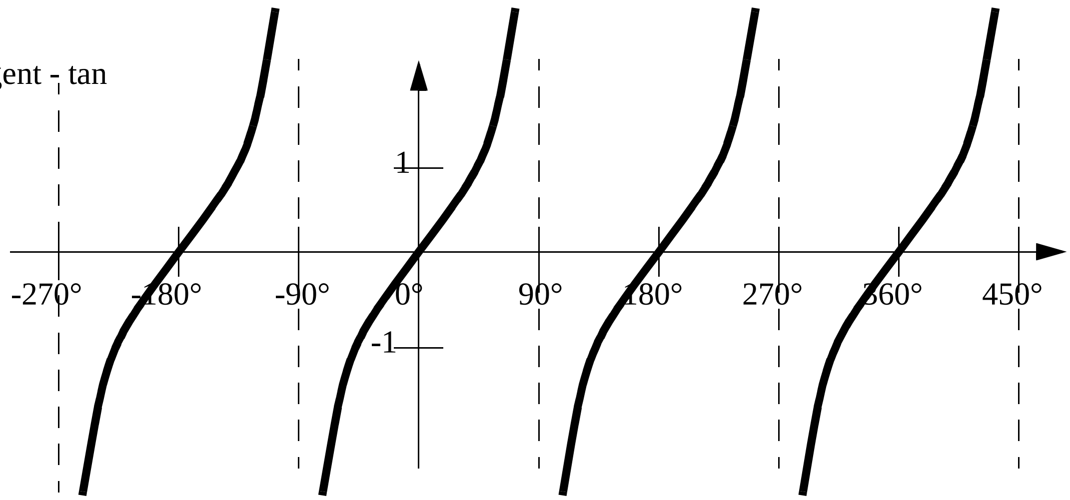
Sine - sin



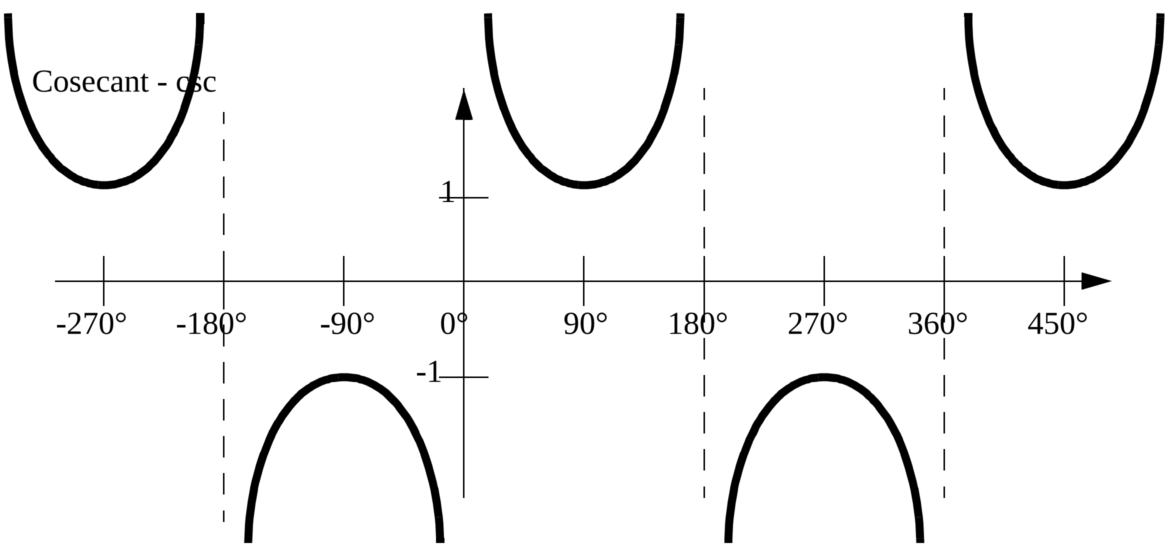
Cosine - cos

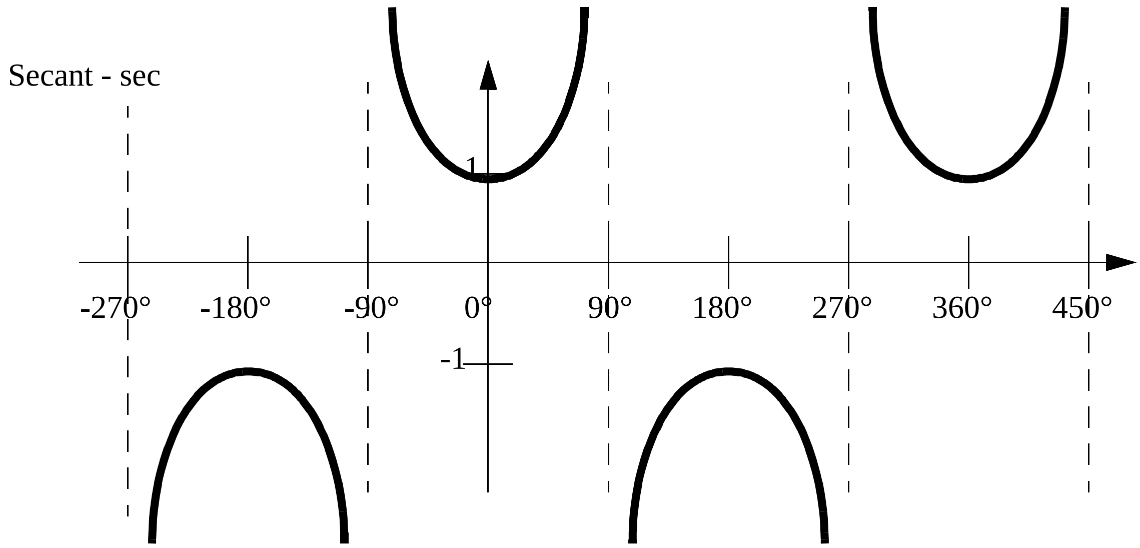


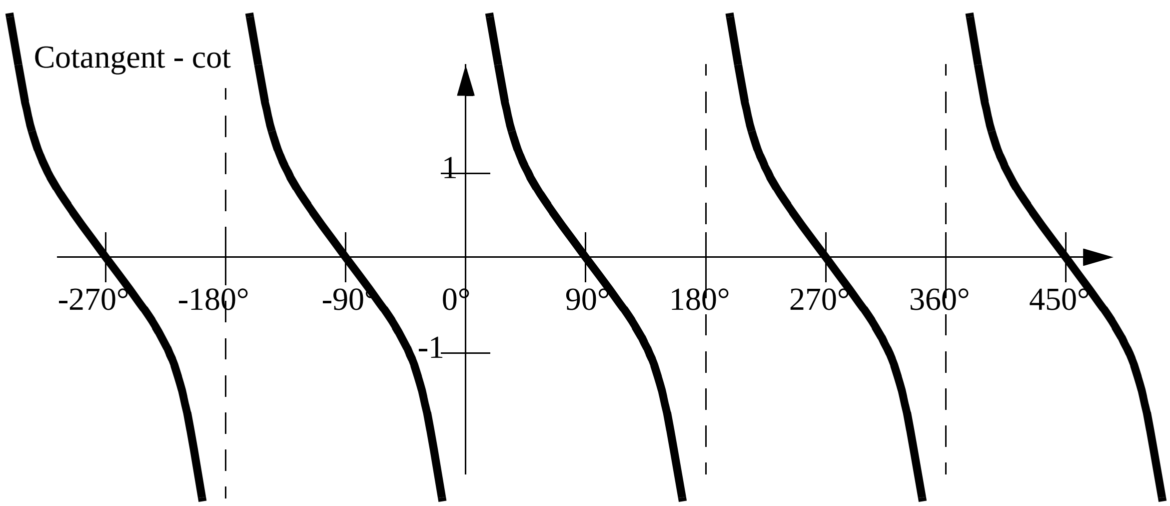
Tangent - tan



Cosecant - csc







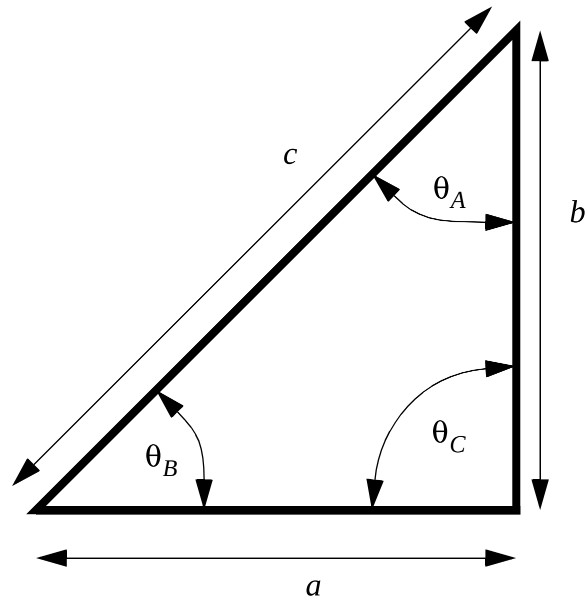
• NOTE: Keep in mind when finding these trig values, that any value that does not lie in the right hand quadrants of cartesian space, may need additions of ±90° or ±180°.

Cosine Law:

$$c^2 = a^2 + b^2 - 2ab\cos\theta_c$$

Sine Law:

$$\frac{a}{\sin\theta_A} = \frac{b}{\sin\theta_B} = \frac{c}{\sin\theta_C}$$



• Now a group of trigonometric relationships will be given. These are often best used when attempting to manipulate equations.

$$\sin(-\theta) = -\sin\theta \qquad \cos(-\theta) = \cos\theta \qquad \tan(-\theta) = -\tan\theta$$

$$\sin\theta = \cos(\theta - 90°) = \cos(90° - \theta) = \text{etc.}$$

$$\sin(\theta_1 \pm \theta_2) = \sin\theta_1 \cos\theta_2 \pm \cos\theta_1 \sin\theta_2 \quad \text{OR} \quad \sin(2\theta) = 2\sin\theta\cos\theta$$

$$\cos(\theta_1 \pm \theta_2) = \cos\theta_1 \cos\theta_2 \mp \sin\theta_1 \sin\theta_2 \quad \text{OR} \quad \cos(2\theta) = (\cos\theta)^2 + (\sin\theta)^2$$

$$\tan(\theta_1 \pm \theta_2) = \frac{\tan\theta_1 \pm \tan\theta_2}{1 \mp \tan\theta_1 \tan\theta_2}$$

$$\cot(\theta_1 \pm \theta_2) = \frac{\cot\theta_1 \cot\theta_2 \mp 1}{\tan\theta_2 \pm \tan\theta_1}$$

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1 - \cos\theta}{2}}$$

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1 + \cos\theta}{2}}$$

-ve if in left hand quadrants

$$\tan\frac{\theta}{2} = \frac{\sin\theta}{1 + \cos\theta} = \frac{1 - \cos\theta}{\sin\theta}$$

$$(\cos\theta)^2 + (\sin\theta)^2 = 1$$

• Numerical values for these functions are given below.

θ (deg)	$\sin\theta$	$\cos\theta$	$\tan\theta$
-90	-1.0	0.0	-infinity
-60	-0.866	0.5	
-45	-0.707	0.707	-1
-30	-0.5	0.866	
0	0	1	0
30	0.5	0.866	
45	0.707	0.707	1
60	0.866	0.5	
90	1.0	0.0	infinity

• These can also be related to complex exponents,

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2} \qquad \sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

34.3.2 Hyperbolic Functions

• The basic definitions are given below,

$$\sinh(x) = \frac{e^{x} - e^{-x}}{2} = \text{hyperbolic sine of x}$$

$$\cosh(x) = \frac{e^{x} + e^{-x}}{2} = \text{hyperbolic cosine of x}$$

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}} = \text{hyperbolic tangent of x}$$

$$\operatorname{csch}(x) = \frac{1}{\sinh(x)} = \frac{2}{e^{x} - e^{-x}} = \text{hyperbolic cosecant of x}$$

$$\operatorname{sech}(x) = \frac{1}{\cosh(x)} = \frac{2}{e^{x} + e^{-x}} = \text{hyperbolic secant of x}$$

$$\coth(x) = \frac{\cosh(x)}{\sinh(x)} = \frac{e^{x} + e^{-x}}{e^{x} - e^{-x}} = \text{hyperbolic cotangent of x}$$

• some of the basic relationships are,

$$\sinh(-x) = -\sinh(x)$$

$$\cosh(-x) = \cosh(x)$$

$$\tanh(-x) = -\tanh(x)$$

$$\operatorname{csch}(-x) = -\operatorname{csch}(x)$$

$$\operatorname{sech}(-x) = \operatorname{sech}(x)$$

$$\coth(-x) = -\coth(x)$$

• Some of the more advanced relationships are,

$$(\cosh x)^2 - (\sinh x)^2 = (\operatorname{sech} x)^2 + (\tanh x)^2 = (\coth x)^2 - (\operatorname{csch} x)^2 = 1$$

$$\sinh(x \pm y) = \sinh(x)\cosh(y) \pm \cosh(x)\sinh(y)$$

$$\cosh(x \pm y) = \cosh(x)\cosh(y) \pm \sinh(x)\sinh(y)$$

$$\tanh(x \pm y) = \frac{\tanh(x) \pm \tanh(y)}{1 \pm \tanh(x)\tanh(y)}$$

• Some of the relationships between the hyperbolic, and normal trigonometry functions are,

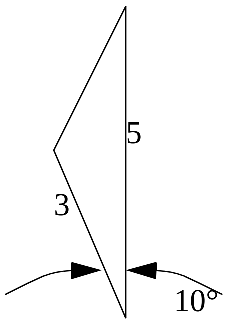
$$\sin(jx) = j\sinh(x) \qquad j\sin(x) = \sinh(jx)$$

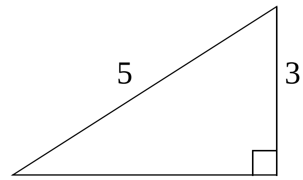
$$\cos(jx) = \cosh(x) \qquad \cos(x) = \cosh(jx)$$

$$\tan(jx) = j\tanh(x) \qquad j\tan(x) = \tanh(jx)$$

34.3.2.1 - Practice Problems

1. Find all of the missing side lengths and corner angles on the two triangles below.





2. Simplify the following expressions.

$$\cos\theta\cos\theta - \sin\theta\sin\theta =$$

$$(s + 3j)(s - 3j)(s + 2j)^2 =$$

(ans.

$$\cos\theta\cos\theta - \sin\theta\sin\theta = \cos(\theta + \theta) = \cos(2\theta)$$

$$(s + 3j)(s - 3j)(s + 2j)^2 = (s^2 - 9j^2)(s^2 + 4js + 4j^2)$$

$$s^4 + 4js^3 + 4j^2s^2 - 9j^2s^2 - 9j^2 4js - 9j^2 4j^2$$

$$s^4 + (4j)s^3 + (5)s^2 + (36j)s + (-36)$$

3. Solve the following partial fraction

$$\frac{4}{x^2 + 3x + 2} =$$

Note: there was a typo here, so an acceptable answer is also. $\frac{1}{x + 0.5}$

(ans.

$$\frac{4}{x^2 + 3x + 2} = \frac{A}{x + 1} + \frac{B}{x + 2} = \frac{Ax + 2A + Bx + B}{x^2 + 3x + 2} = \frac{(2A + B) + (A + B)x}{x^2 + 3x + 2}$$

$$A + B = 0 \qquad A = -B$$

$$2A + B = 4 = -2B + B = -B \qquad B = -4 \quad A = 4 \qquad \frac{4}{x + 1} + \frac{-4}{x + 2}$$

34.3.3 Geometry

• A set of the basic 2D and 3D geometric primitives are given, and the notation used is described below,

A = contained area

P = perimeter distance

V = contained volume

S = surface area

x, y, z = centre of mass

$\bar{x}, \bar{y}, \bar{z}$ = centroid

I_x, I_y, I_z = moment of inertia of area (or second moment of inertia)

AREA PROPERTIES:

$$I_x = \int_A y^2 dA = \text{the second moment of inertia about the y-axis}$$

$$I_y = \int_A x^2 dA = \text{the second moment of inertia about the x-axis}$$

$$I_{xy} = \int_A xy dA = \text{the product of inertia}$$

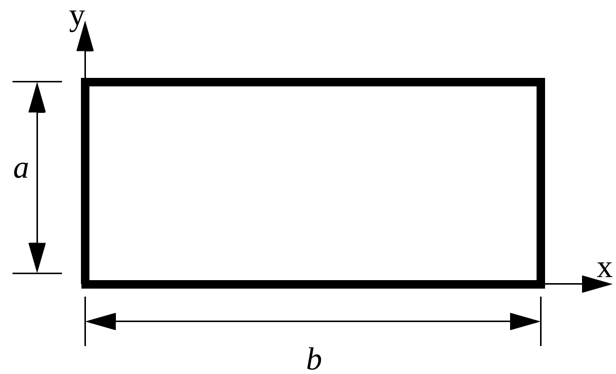
$$J_O = \int_A r^2 dA = \int_A (x^2 + y^2) dA = I_x + I_y = \text{The polar moment of inertia}$$

$$\bar{x} = \frac{\int_A x dA}{\int_A dA} = \text{centroid location along the x-axis}$$

$$\bar{y} = \frac{\int_A y dA}{\int_A dA} = \text{centroid location along the y-axis}$$

Rectangle/Square:

$$A = ab$$
$$P = 2a + 2b$$

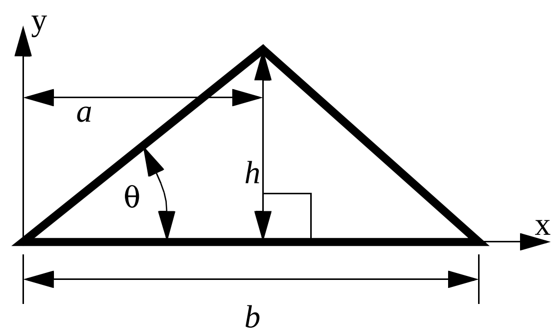


Centroid:	Moment of Inertia (about centroid axes):	Moment of Inertia (about origin axes):
$\bar{x} = \frac{b}{2}$	$\bar{I}_x = \frac{ba^3}{12}$	$I_x = \frac{ba^3}{3}$
$\bar{y} = \frac{a}{2}$	$\bar{I}_y = \frac{b^3a}{12}$	$I_y = \frac{b^3a}{3}$
	$\overline{I_{xy}} = 0$	$I_{xy} = \frac{b^2a^2}{4}$

Triangle:

$$A = \frac{bh}{2}$$

$$P =$$



Centroid:

$$\bar{x} = \frac{a+b}{3}$$

$$\bar{y} = \frac{h}{3}$$

Moment of Inertia (about centroid axes):

$$\bar{I}_x = \frac{bh^3}{36}$$

$$\bar{I}_y = \frac{bh}{36}(a^2 + b^2 - ab)$$

$$\overline{I_{xy}} = \frac{bh^2}{72}(2a - b)$$

Moment of Inertia (about origin axes):

$$I_x = \frac{bh^3}{12}$$

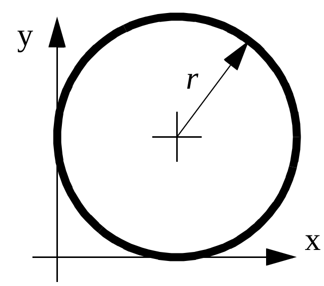
$$I_y = \frac{bh}{12}(a^2 + b^2 - ab)$$

$$I_{xy} = \frac{bh^2}{24}(2a - b)$$

Circle:

$$A = \pi r^2$$

$$P = 2\pi r$$



Centroid:

$$\bar{x} = r$$

$$\bar{y} = r$$

Moment of Inertia (about centroid axes):

$$\bar{I}_x = \frac{\pi r^4}{4}$$

$$\bar{I}_y = \frac{\pi r^4}{4}$$

$$\overline{I_{xy}} = 0$$

Moment of Inertia (about origin axes):

$$I_x =$$

$$I_y =$$

$$I_{xy} =$$

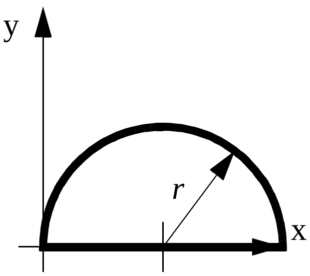
Mass Moment of Inertia (about centroid):

$$J_z = \frac{Mr^2}{2}$$

Half Circle:

$$A = \frac{\pi r^2}{2}$$

$$P = \pi r + 2r$$



Centroid:

$$\bar{x} = r$$

$$\bar{y} = \frac{4r}{3\pi}$$

Moment of Inertia
(about centroid axes):

$$\bar{I}_x = \left(\frac{\pi}{8} - \frac{8}{9\pi}\right) r^4$$

$$\bar{I}_y = \frac{\pi r^4}{8}$$

$$I_{xy} = 0$$

Moment of Inertia
(about origin axes):

$$I_x = \frac{\pi r^4}{8}$$

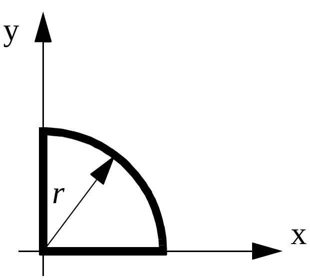
$$I_y = \frac{\pi r^4}{8}$$

$$I_{xy} = 0$$

Quarter Circle:

$$A = \frac{\pi r^2}{4}$$

$$P = \frac{\pi r}{2} + 2r$$



Centroid:

$$\bar{x} = \frac{4r}{3\pi}$$

$$\bar{y} = \frac{4r}{3\pi}$$

Moment of Inertia
(about centroid axes):

$$\bar{I}_x = 0.05488 r^4$$

$$\bar{I}_y = 0.05488 r^4$$

$$\overline{I_{xy}} = -0.01647 r^4$$

Moment of Inertia
(about origin axes):

$$I_x = \frac{\pi r^4}{16}$$

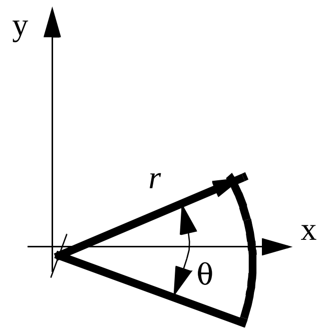
$$I_y = \frac{\pi r^4}{16}$$

$$I_{xy} = \frac{r^4}{8}$$

Circular Arc:

$$A = \frac{\theta r^2}{2}$$

$$P = \theta r + 2r$$



Centroid:

$$\bar{x} = \frac{2r\sin\frac{\theta}{2}}{3\theta}$$

$$\bar{y} = 0$$

Moment of Inertia
(about centroid axes):

$$\bar{I}_x =$$

$$\bar{I}_y =$$

$$\overline{I_{xy}} =$$

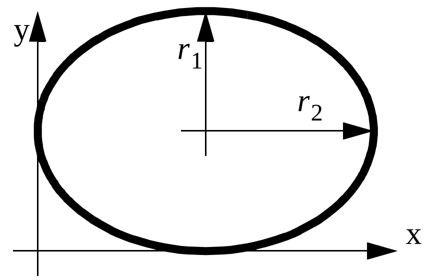
Moment of Inertia
(about origin axes):

$$I_x = \frac{r^4}{8}(\theta - \sin\theta)$$

$$I_y = \frac{r^4}{8}(\theta + \sin\theta)$$

$$I_{xy} = 0$$

Ellipse:



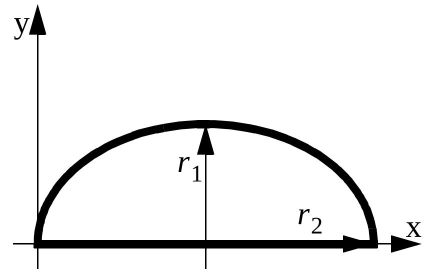
$$A = \pi r_1 r_2$$

$$P = 4r_1 \int_0^{\frac{\pi}{2}} \sqrt{1 - \frac{\sqrt{r_1^2 + r_2^2}}{a}(\sin\theta)^2} d\theta$$

$$P \approx 2\pi\sqrt{\frac{r_1^2 + r_2^2}{2}}$$

Centroid:	Moment of Inertia (about centroid axes):	Moment of Inertia (about origin axes):
$\bar{x} = r_2$	$\bar{I}_x = \frac{\pi r_1^3 r_2}{4}$	$I_x =$
$\bar{y} = r_1$	$\bar{I}_y = \frac{\pi r_1 r_2^3}{4}$	$I_y =$
	$\overline{I_{xy}} =$	$I_{xy} =$

Half Ellipse:



$$A = \frac{\pi r_1 r_2}{2}$$

$$P = 2r_1\int_0^{\frac{\pi}{2}}\sqrt{1 - \frac{\sqrt{r_1^2 + r_2^2}}{a}(\sin\theta)^2}d\theta + 2r_2$$

$$P \approx \pi\sqrt{\frac{r_1^2 + r_2^2}{2}} + 2r_2$$

Centroid:

$$\bar{x} = r_2$$

$$\bar{y} = \frac{4r_1}{3\pi}$$

Moment of Inertia (about centroid axes):

$$\overline{I_x} = 0.05488 r_2 r_1^3$$

$$\overline{I_y} = 0.05488 r_2^3 r_1$$

$$\overline{I_{xy}} = -0.01647 r_1^2 r_2^2$$

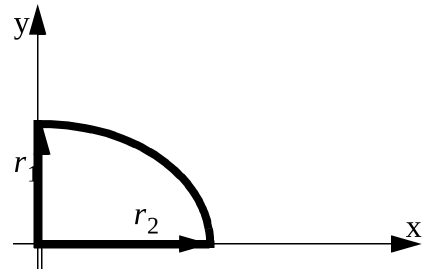
Moment of Inertia (about origin axes):

$$I_x = \frac{\pi r_2 r_1^3}{16}$$

$$I_y = \frac{\pi r_2^3 r_1}{16}$$

$$I_{xy} = \frac{r_1^2 r_2^2}{8}$$

Quarter Ellipse:



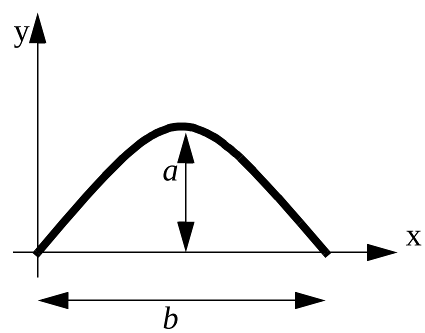
$$A = \frac{\pi r_1 r_2}{4}$$

$$P = r_1 \int_0^{\frac{\pi}{2}} \sqrt{1 - \frac{\sqrt{r_1^2 + r_2^2}}{a}(\sin\theta)^2} d\theta + 2r_2$$

$$P \approx \frac{\pi}{2}\sqrt{\frac{r_1^2 + r_2^2}{2}} + 2r_2$$

Centroid:	Moment of Inertia (about centroid axes):	Moment of Inertia (about origin axes):
$\bar{x} = \frac{4r_2}{3\pi}$	$\bar{I}_x =$	$I_x = \pi r_2 r_1^3$
$\bar{y} = \frac{4r_1}{3\pi}$	$\bar{I}_y =$	$I_y = \pi r_2^3 r_1$
	$\overline{I_{xy}} =$	$I_{xy} = \frac{r_2^2 r_1^2}{8}$

Parabola:



$$A = \frac{2}{3}ab$$

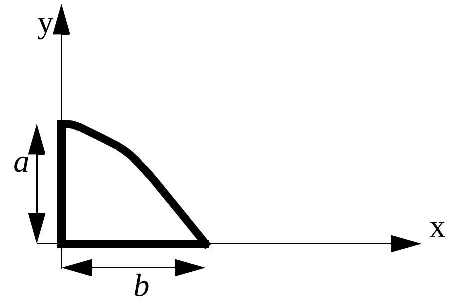
$$P = \frac{\sqrt{b^2 + 16a^2}}{2} + \frac{b^2}{8a}\ln\left(\frac{4a + \sqrt{b^2 + 16a^2}}{b}\right)$$

Centroid:	Moment of Inertia (about centroid axes):	Moment of Inertia (about origin axes):
$\bar{x} = \frac{b}{2}$	$\bar{I}_x =$	$I_x =$
$\bar{y} = \frac{2a}{5}$	$\bar{I}_y =$	$I_y =$
	$\overline{I_{xy}} =$	$I_{xy} =$

Half Parabola:

$$A = \frac{ab}{3}$$

$$P = \frac{\sqrt{b^2 + 16a^2}}{4} + \frac{b^2}{16a}\ln\left(\frac{4a + \sqrt{b^2 + 16a^2}}{b}\right)$$



Centroid:

$$\bar{x} = \frac{3b}{8}$$

$$\bar{y} = \frac{2a}{5}$$

Moment of Inertia (about centroid axes):

$$\overline{I_x} = \frac{8ba^3}{175}$$

$$\overline{I_y} = \frac{19b^3a}{480}$$

$$\overline{I_{xy}} = \frac{b^2a^2}{60}$$

Moment of Inertia (about origin axes):

$$I_x = \frac{2ba^3}{7}$$

$$I_y = \frac{2b^3a}{15}$$

$$I_{xy} = \frac{b^2a^2}{6}$$

• A general class of geometries are conics. This for is shown below, and can be used to represent many of the simple shapes represented by a polynomial.

$$Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0$$

Conditions		
	$A = B = C = 0$	straight line
	$B = 0, A = C$	circle
	$B^2 - AC < 0$	ellipse
	$B^2 - AC = 0$	parabola
	$B^2 - AC > 0$	hyperbola

VOLUME PROPERTIES:

$$I_x = \int_V r_x^2 dV = \text{the moment of inertia about the x-axis}$$

$$I_y = \int_V r_y^2 dV = \text{the moment of inertia about the y-axis}$$

$$I_z = \int_V r_z^2 dV = \text{the moment of inertia about the z-axis}$$

$$\bar{x} = \frac{\int_V x dV}{\int_V dV} = \text{centroid location along the x-axis}$$

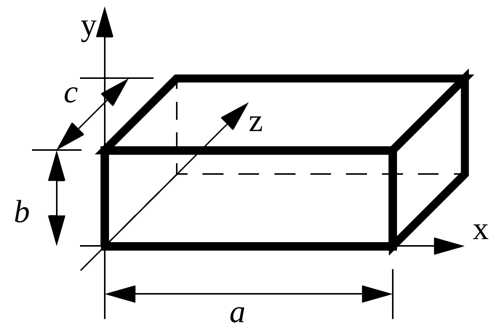
$$\bar{y} = \frac{\int_V y dV}{\int_V dV} = \text{centroid location along the y-axis}$$

$$\bar{z} = \frac{\int_V z dV}{\int_V dV} = \text{centroid location along the z-axis}$$

Parallelepiped (box):

$$V = abc$$

$$S = 2(ab + ac + bc)$$

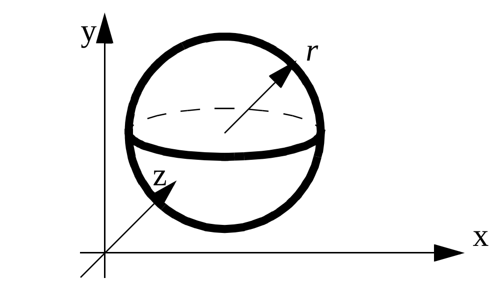


Centroid:	Moment of Inertia (about centroid axes):	Moment of Inertia (about origin axes):	Mass Moment of Inertia (about centroid):
$\bar{x} = \frac{a}{2}$	$\bar{I}_x = \frac{M(a^2 + b^2)}{12}$	$I_x =$	$J_x =$
$\bar{y} = \frac{b}{2}$	$\bar{I}_y = \frac{M(a^2 + c^2)}{12}$	$I_y =$	$J_y =$
$\bar{z} = \frac{c}{2}$	$\bar{I}_z = \frac{M(b^2 + a^2)}{12}$	$I_z =$	$J_z =$

Sphere:

$$V = \frac{4}{3}\pi r^3$$

$$S = 4\pi r^2$$

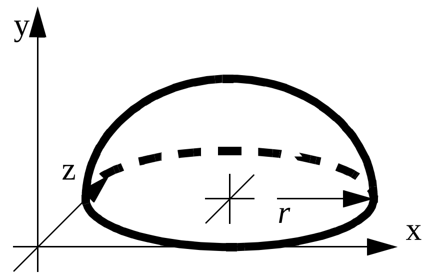


Centroid:	Moment of Inertia (about centroid axes):	Moment of Inertia (about origin axes):	Mass Moment of Inertia (about centroid):
$\bar{x} = r$	$\bar{I}_x = \frac{2Mr^2}{5}$	$I_x =$	$J_x = \frac{2Mr^2}{5}$
$\bar{y} = r$	$\bar{I}_y = \frac{2Mr^2}{5}$	$I_y =$	$J_y = \frac{2Mr^2}{5}$
$\bar{z} = r$	$\bar{I}_z = \frac{2Mr^2}{5}$	$I_z =$	$J_z = \frac{2Mr^2}{5}$

Hemisphere:

$$V = \frac{2}{3}\pi r^3$$

$$S =$$



Centroid:

$\bar{x} = r$

$\bar{y} = \frac{3r}{8}$

$\bar{z} = r$

Moment of Inertia (about centroid axes):

$\bar{I}_x = \frac{83}{320}Mr^2$

$\bar{I}_y = \frac{2Mr^2}{5}$

$\bar{I}_z = \frac{83}{320}Mr^2$

Moment of Inertia (about origin axes):

$I_x =$

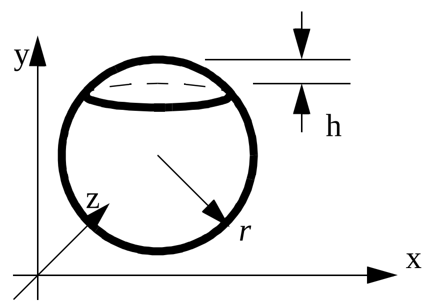
$I_y =$

$I_z =$

Cap of a Sphere:

$$V = \frac{1}{3}\pi h^2(3r - h)$$

$$S = 2\pi rh$$



Centroid:

$\bar{x} = r$

$\bar{y} =$

$\bar{z} = r$

Moment of Inertia (about centroid axes):

$\bar{I}_x =$

$\bar{I}_y =$

$\bar{I}_z =$

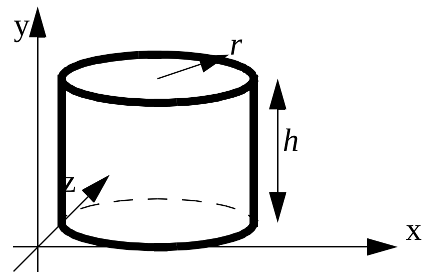
Moment of Inertia (about origin axes):

$I_x =$

$I_y =$

$I_z =$

Cylinder:

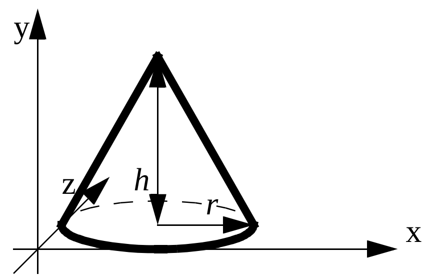


$$V = h\pi r^2$$

$$S = 2\pi rh + 2\pi r^2$$

Centroid:	Moment of Inertia (about centroid axis):	Moment of Inertia (about origin axis):	Mass Moment of Inertia (about centroid):
$\bar{x} = r$	$\bar{I}_x = M\left(\frac{h^2}{12} + \frac{r^2}{4}\right)$	$I_x = M\left(\frac{h^2}{3} + \frac{r^2}{4}\right)$	$J_x = \frac{M(3r^2 + h^2)}{12}$
$\bar{y} = \frac{h}{2}$	$\bar{I}_y = \frac{Mr^2}{2}$	$I_y =$	$J_y = \frac{Mr^2}{2}$
$\bar{z} = r$	$\bar{I}_z = M\left(\frac{h^2}{12} + \frac{r^2}{4}\right)$	$I_z = M\left(\frac{h^2}{3} + \frac{r^2}{4}\right)$	$J_z = \frac{M(3r^2 + h^2)}{12}$

Cone:



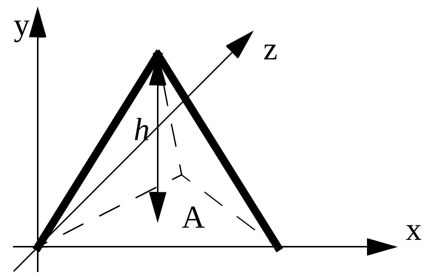
$$V = \frac{1}{3}\pi r^2 h$$

$$S = \pi r\sqrt{r^2 + h^2}$$

Centroid:	Moment of Inertia (about centroid axes):	Moment of Inertia (about origin axes):
$\bar{x} = r$	$\bar{I}_x = M\left(\frac{3h^3}{80} + \frac{3r^2}{20}\right)$	$I_x =$
$\bar{y} = \frac{h}{4}$	$\bar{I}_y = \frac{3Mr^2}{10}$	$I_y =$
$\bar{z} = r$	$\bar{I}_z = M\left(\frac{3h^3}{80} + \frac{3r^2}{20}\right)$	$I_z =$

Tetrahedron:

$$V = \frac{1}{3}Ah$$

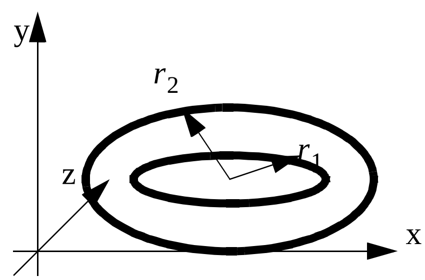


Centroid:	Moment of Inertia (about centroid axes):	Moment of Inertia (about origin axes):
$\bar{x} =$	$\bar{I}_x =$	$I_x =$
$\bar{y} = \frac{h}{4}$	$\bar{I}_y =$	$I_y =$
$\bar{z} =$	$\bar{I}_z =$	$I_z =$

Torus:

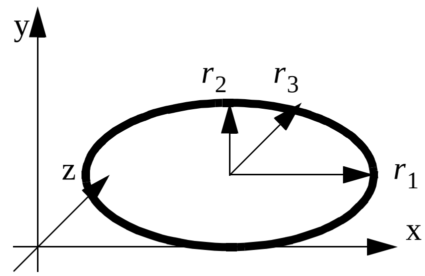
$$V = \frac{1}{4}\pi^2(r_1 + r_2)(r_2 - r_1)^2$$

$$S = \pi^2(r_2^2 - r_1^2)$$



Centroid:	Moment of Inertia (about centroid axes):	Moment of Inertia (about origin axes):
$\bar{x} = r_2$	$\bar{I}_x =$	$I_x =$
$\bar{y} = \left(\frac{r_2 - r_1}{2}\right)$	$\bar{I}_y =$	$I_y =$
$\bar{z} = r_2$	$\bar{I}_z =$	$I_z =$

Ellipsoid:

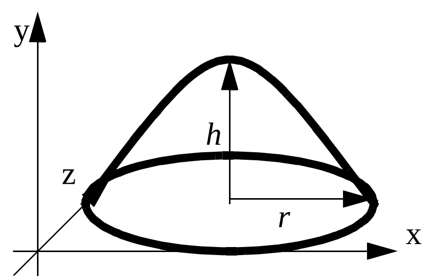


$$V = \frac{4}{3}\pi r_1 r_2 r_3$$

$$S =$$

Centroid:	Moment of Inertia (about centroid axes):	Moment of Inertia (about origin axes):
$\bar{x} = r_1$	$\bar{I}_x =$	$I_x =$
$\bar{y} = r_2$	$\bar{I}_y =$	$I_y =$
$\bar{z} = r_3$	$\bar{I}_z =$	$I_z =$

Paraboloid:



$$V = \frac{1}{2}\pi r^2 h$$

$$S =$$

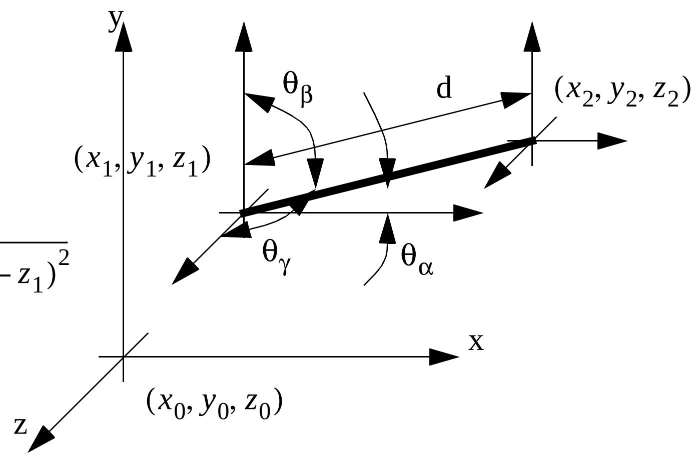
Centroid:	Moment of Inertia (about centroid axes):	Moment of Inertia (about origin axes):
$\bar{x} = r$	$\bar{I}_x =$	$I_x =$
$\bar{y} =$	$\bar{I}_y =$	$I_y =$
$\bar{z} = r$	$\bar{I}_z =$	$I_z =$

34.3.4 Planes, Lines, etc.

• The most fundamental mathematical geometry is a line. The basic relationships are given below,

$y = mx + b$	defined with a slope and intercept
$m_{\text{perpendicular}} = \frac{1}{m}$	a slope perpendicular to a line
$m = \frac{y_2 - y_1}{x_2 - x_1}$	the slope using two points
$\frac{x}{a} + \frac{y}{b} = 1$	as defined by two intercepts

• If we assume a line is between two points in space, and that at one end we have a local reference frame, there are some basic relationships that can be derived.



$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

The direction cosines of the angles are,

$$\theta_\alpha = \operatorname{acos}\left(\frac{x_2 - x_1}{d}\right) \qquad \theta_\beta = \operatorname{acos}\left(\frac{y_2 - y_1}{d}\right) \qquad \theta_\gamma = \operatorname{acos}\left(\frac{z_2 - z_1}{d}\right)$$

$$(\cos\theta_\alpha)^2 + (\cos\theta_\beta)^2 + (\cos\theta_\gamma)^2 = 1$$

The equation of the line is,

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$ Explicit

$$(x, y, z) = (x_1, y_1, z_1) + t((x_2, y_2, z_2) - (x_1, y_1, z_1))$$ Parametric t=[0,1]

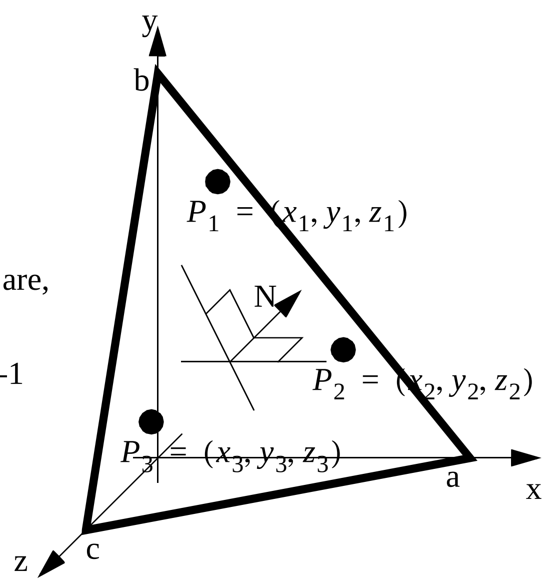
• The relationships for a plane are,

The explicit equation for a plane is,

$$Ax + By + Cz + D = 0$$

where the coefficients defined by the intercepts are,

$$A = \frac{1}{a} \quad B = \frac{1}{b} \quad C = \frac{1}{c} \quad D = -1$$



The determinant can also be used,

$$det\begin{bmatrix} x-x_1 & y-y_1 & z-z_1 \\ x-x_2 & y-y_2 & z-z_2 \\ x-x_3 & y-y_3 & z-z_3 \end{bmatrix} = 0$$

$$\therefore det\begin{bmatrix} y_2-y_1 & z_2-z_1 \\ y_3-y_1 & z_3-z_1 \end{bmatrix}(x-x_1) + det\begin{bmatrix} z_2-z_1 & x_2-x_1 \\ z_3-z_1 & x_3-x_1 \end{bmatrix}(y-y_1)$$

$$+ det\begin{bmatrix} x_2-x_1 & y_2-y_1 \\ x_3-x_1 & y_3-y_1 \end{bmatrix}(z-z_1) = 0$$

The normal to the plane (through the origin) is,

$$(x, y, z) = t(A, B, C)$$

34.3.5 Practice Problems

1. What is the circumferenece of a circle? What is the area? What is the ratio of the area to the circumference?

2. What is the equation of a line that passes through the points below?

a) (0, 0) to (2, 2)
b) (1, 0) to (0, 1)
c) (3, 4) to (2. 9)

3. Find a line that is perpendicular to the line through the points (2, 1) and (1, 2). The perpendicular line passes through (3, 5).

4. Manipulate the following equations to solve for ‘x’.

a) $x^2 + 3x = -2$
b) $\sin x = \cos x$

(ans.

a) $x^2 + 3x = -2$

$x^2 + 3x + 2 = 0$

$$x = \frac{-3 \pm \sqrt{3^2 - 4(1)(2)}}{2(1)} = \frac{-3 \pm \sqrt{9-8}}{2} = \frac{-3 \pm 1}{2} = -1, -2$$

b) $\sin x = \cos x$

$$\frac{\sin x}{\cos x} = 1$$

$\tan x = 1$

$x = \operatorname{atan} 1$

$x = \ldots, -135°, 45°, 225°, \ldots$

5. Simplify the following expressions.

a) $$\sin 2\theta\left(\frac{(\cos 2\theta)^2}{\sin 2\theta} + \sin 2\theta\right)$$

(ans.

a) $$\sin 2\theta\left(\frac{(\cos 2\theta)^2}{\sin 2\theta} + \sin 2\theta\right) = (\cos 2\theta)^2 + (\sin 2\theta)^2 = 1$$

6. A line that passes through the point (1, 2) and has a slope of 2. Find the equation for the line, and for a line perpendicular to it.

(ans. given,

$$y = mx + b$$
$$m = 2$$
$$x = 1$$
$$y = 2$$

substituting

$$2 = 2(1) + b \qquad b = 0$$
$$y = 2x$$

perpendicular

$$m_{perp} = -\frac{1}{m} = -0.5$$
$$y = -0.5x$$

34.4 COORDINATE SYSTEMS

34.4.1 Complex Numbers

• In this section, as in all others, ‘j’ will be the preferred notation for the complex number, this is to help minimize confusion with the ‘i’ used for current in electrical engineering.

• The basic algebraic properties of these numbers are, The Complex Number:

$$j = \sqrt{-1} \qquad j^2 = -1$$

Complex Numbers:

$$a + bj$$ where, a and b are both real numbers

Complex Conjugates (denoted by adding an asterisk '*' the variable):

$$N = a + bj \qquad N* = a - bj$$

Basic Properties:

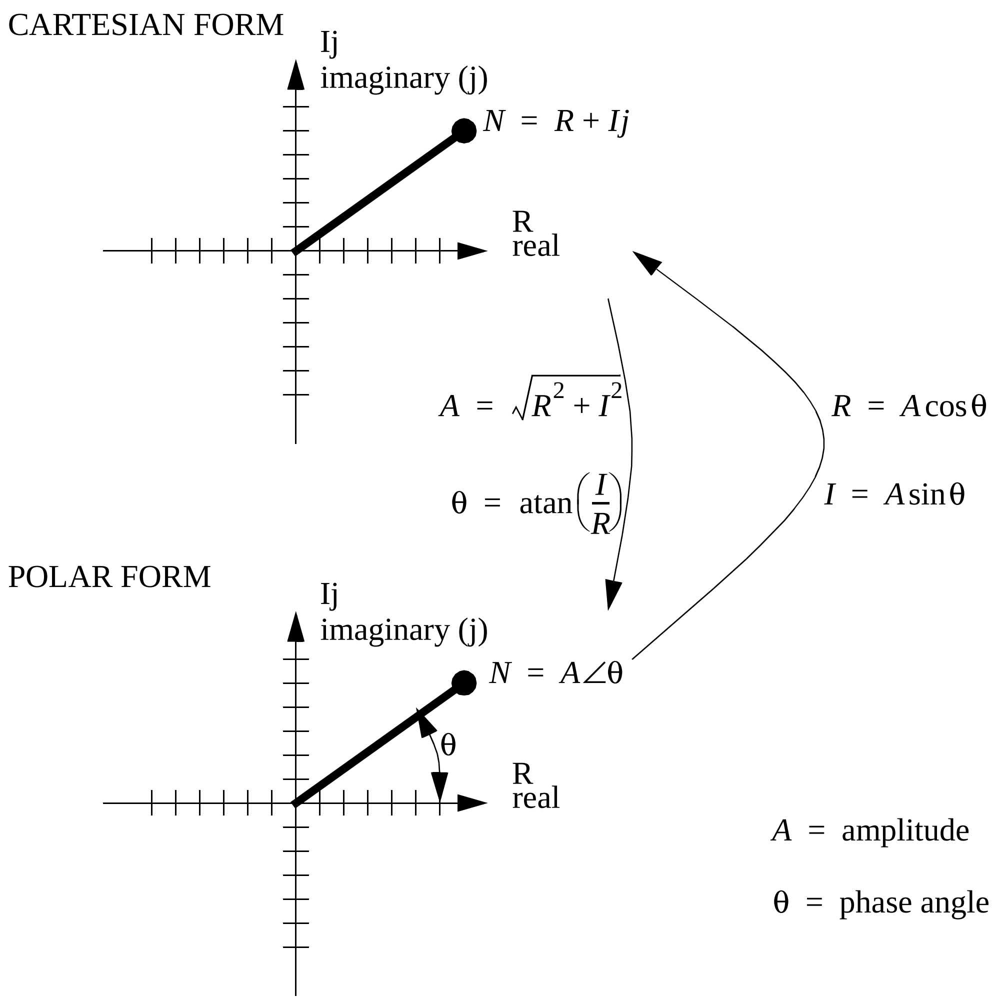
$$(a + bj) + (c + dj) = (a + c) + (b + d)j$$

$$(a + bj) - (c + dj) = (a - c) + (b - d)j$$

$$(a + bj) \cdot (c + dj) = (ac - bd) + (ad + bc)j$$

$$\frac{N}{M} = \frac{a + bj}{c + dj} = \frac{N}{M}\left(\frac{N*}{N*}\right) = \left(\frac{a + bj}{c + dj}\right)\left(\frac{c - dj}{c - dj}\right) = \frac{ac + bd}{c^2 + d^2} + \left(\frac{bc - ad}{c^2 + d^2}\right)j$$

• We can also show complex numbers graphically. These representations lead to alternative representations. If it in not obvious above, please consider the notation above uses a cartesian notation, but a polar notation can also be very useful when doing large calculations.



• We can also do calculations using polar notation (this is well suited to multiplication and division, whereas cartesian notation is easier for addition and subtraction),

$$A\angle\theta = A(\cos\theta + j\sin\theta) = Ae^{j\theta}$$

$$(A_1\angle\theta_1)(A_2\angle\theta_2) = (A_1A_2)\angle(\theta_1 + \theta_2)$$

$$\frac{(A_1\angle\theta_1)}{(A_2\angle\theta_2)} = \left(\frac{A_1}{A_2}\right)\angle(\theta_1 - \theta_2)$$

$$(A\angle\theta)^n = (A^n)\angle(n\theta) \qquad \text{(DeMoivre's theorem)}$$

• Note that DeMoivre's theorem can be used to find exponents (including roots) of complex numbers

• Euler's formula: $e^{j\theta} = \cos\theta + j\sin\theta$

• From the above, the following useful identities arise:

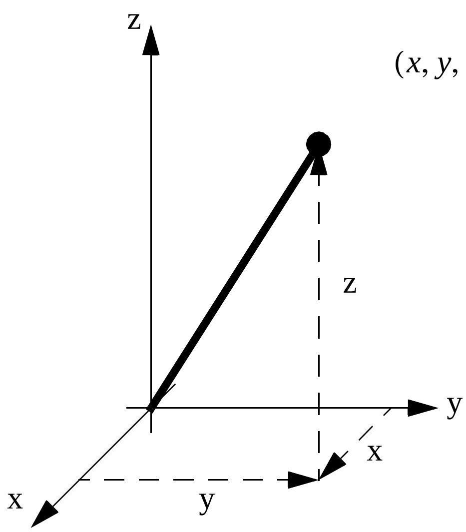
$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

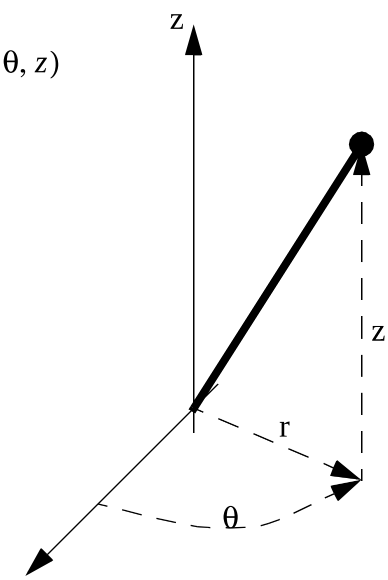
$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

34.4.2 Cylindrical Coordinates

• Basically, these coordinates appear as if the cartesian box has been replaced with a cylinder,

$$(x, y, z) \leftrightarrow (r, \theta, z)$$





$$x = r\cos\theta$$

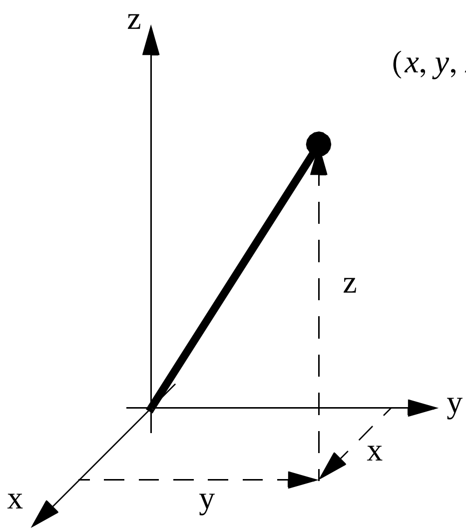
$$y = r\sin\theta$$

$$r = \sqrt{x^2 + y^2}$$

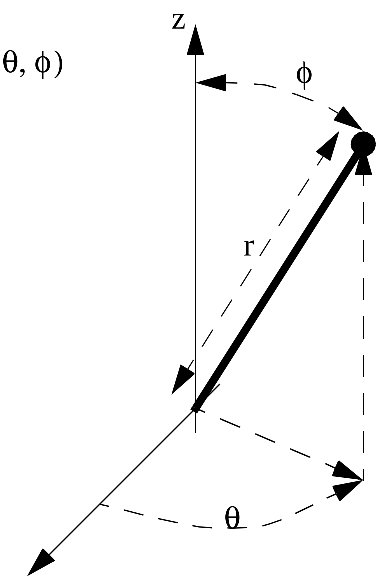
$$\theta = \operatorname{atan}\left(\frac{y}{x}\right)$$

34.4.3 Spherical Coordinates

• This system replaces the cartesian box with a sphere,



$(x, y, z) \leftrightarrow (r, \theta, \phi)$



$$x = r\sin\theta\cos\phi$$

$$y = r\sin\theta\sin\phi$$

$$z = r\cos\theta$$

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\theta = \operatorname{atan}\left(\frac{y}{x}\right)$$

$$\phi = \operatorname{acos}\left(\frac{z}{r}\right)$$

34.4.4 Practice Problems

1. Simplify the following expressions.

a) $\frac{16}{(4j + 4)^2}$

b) $\frac{3j + 5}{(4j + 3)^2}$

c) $(3 + 5j)4j$ where, $j = \sqrt{-1}$

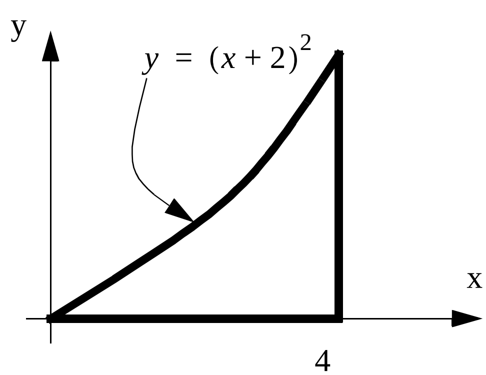
(ans. a) $$\frac{16}{(4j+4)^2} = \frac{16}{-16+32j+16} = \frac{16}{32j} = -0.5j$$

b) $$\frac{3j+5}{(4j+3)^2} = \frac{3j+5}{-16+24j+9} = \frac{3j+5}{-7+24j}\left(\frac{-7-24j}{-7-24j}\right) = \frac{-35-141j+72}{49+576} = \frac{37-141j}{625}$$

c) $$(3+5j)4j = 12j+20j^2 = 12j-20$$

2. For the shape defined below,

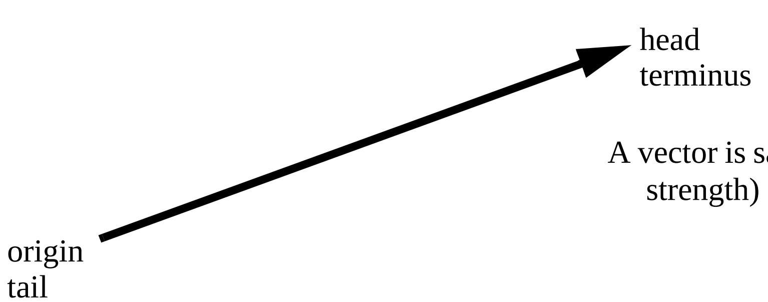
a) find the area of the shape.
b) find the centroid of the shape.
c) find the moment of inertia of the shape about the centroid.



34.5 MATRICES AND VECTORS

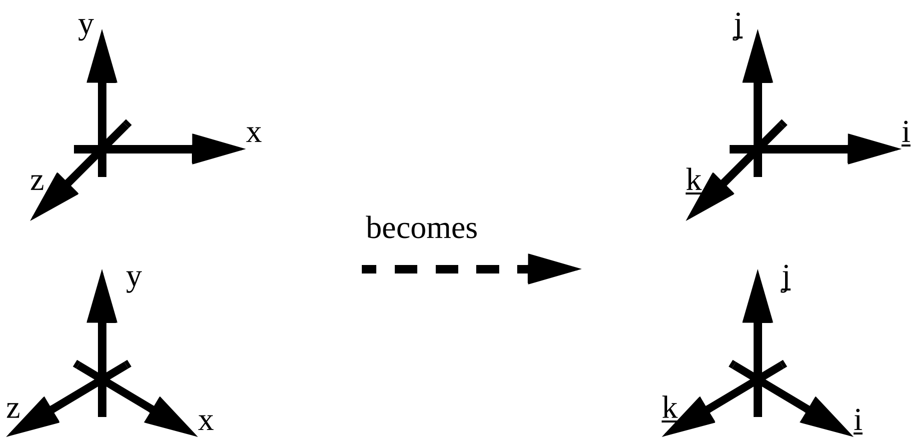
34.5.1 Vectors

• Vectors are often drawn with arrows, as shown below,



A vector is said to have magnitude (length or strength) and direction.

• Cartesian notation is also a common form of usage.

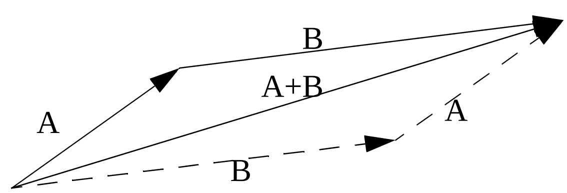


• Vectors can be added and subtracted, numerically and graphically,

$$A = (2, 3, 4) \qquad A + B = (2 + 7, 3 + 8, 4 + 9)$$

$$B = (7, 8, 9) \qquad A - B = (2 - 7, 3 - 8, 4 - 9)$$

Parallelogram Law



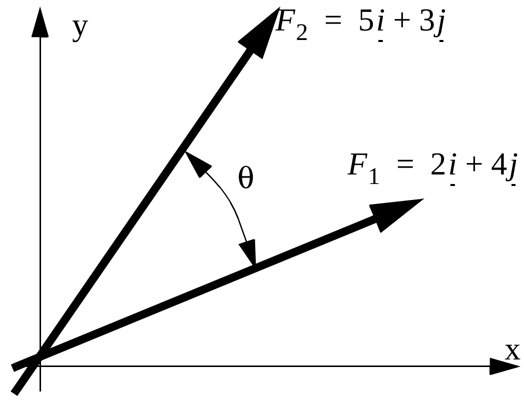
34.5.2 Dot (Scalar) Product

• We can use a dot product to find the angle between two vectors

$$\cos\theta = \frac{F_1 \bullet F_2}{|F_1||F_2|}$$

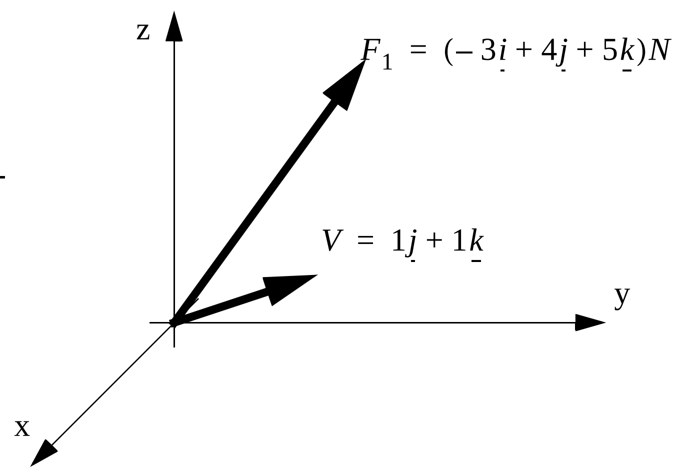
$$\therefore \theta = \operatorname{acos}\left(\frac{(2)(5) + (4)(3)}{\sqrt{2^2 + 4^2}\sqrt{5^2 + 3^2}}\right)$$

$$\therefore \theta = \operatorname{acos}\left(\frac{22}{(4.47)(6)}\right) = 32.5°$$



• We can use a dot product to project one vector onto another vector.

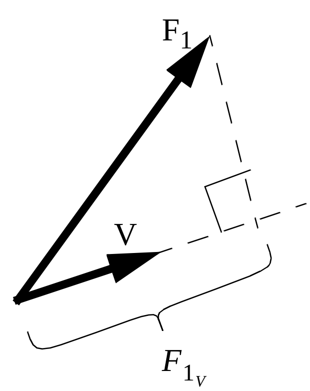
We want to find the component of force F_1 that projects onto the vector V. To do this we first convert V to a unit vector, if we do not, the component we find will be multiplied by the magnitude of V.



$$\lambda_V = \frac{V}{|V|} = \frac{1\underline{j} + 1\underline{k}}{\sqrt{1^2 + 1^2}} = 0.707\underline{j} + 0.707\underline{k}$$

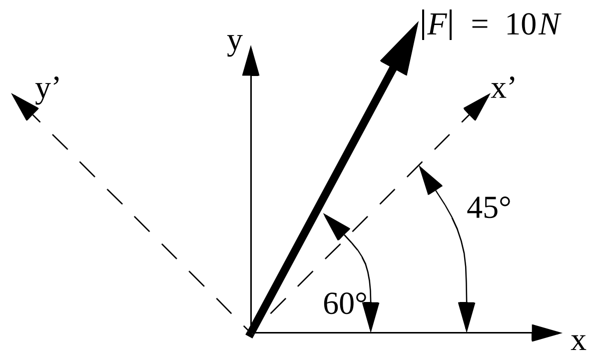
$$F_{1_V} = \lambda_V \bullet F_1 = (0.707\underline{j} + 0.707\underline{k}) \bullet (-3\underline{i} + 4\underline{j} + 5\underline{k})N$$

$$\therefore F_{1_V} = (0)(-3) + (0.707)(4) + (0.707)(5) = 6N$$



• We can consider the basic properties of the dot product and units vectors.

Unit vectors are useful when breaking up vector magnitudes and direction. As an example consider the vector, and the displaced x-y axes shown below as x'-y'.



We could write out 5 vectors here, relative to the x-y axis,

$$\text{x axis} = 2\underline{i}$$
$$\text{y axis} = 3\underline{j}$$
$$\text{x' axis} = 1\underline{i} + 1\underline{j}$$
$$\text{y' axis} = -1\underline{i} + 1\underline{j}$$
$$F = 10N\angle 60° = (10\cos 60°)\underline{i} + (10\sin 60°)\underline{j}$$

None of these vectors has a magnitude of 1, and hence they are not unit vectors. But, if we find the equivalent vectors with a magnitude of one we can simplify many tasks. In particular if we want to find the x and y components of F relative to the x-y axis we can use the dot product.

$$\lambda_x = 1\underline{i} + 0\underline{j} \quad \text{(unit vector for the x-axis)}$$

$$F_x = \lambda_x \bullet F = (1\underline{i} + 0\underline{j}) \bullet [(10\cos 60°)\underline{i} + (10\sin 60°)\underline{j}]$$
$$\therefore \quad = (1)(10\cos 60°) + (0)(10\sin 60°) = 10N\cos 60°$$

This result is obvious, but consider the other obvious case where we want to project a vector onto itself,

$$\lambda_F = \frac{F}{|F|} = \frac{10\cos 60° i + 10\sin 60° j}{10} = \cos 60° i + \sin 60° j$$

Incorrect - Not using a unit vector

$$F_F = F \bullet F$$
$$= ((10\cos 60°)i + (10\sin 60°)j) \bullet ((10\cos 60°)i + (10\sin 60°)j)$$
$$= (10\cos 60°)(10\cos 60°) + (10\sin 60°)(10\sin 60°)$$
$$= 100((\cos 60°)^2 + (\sin 60°)^2) = \cancel{100}$$

Using a unit vector

$$F_F = F \bullet \lambda_F$$
$$= ((10\cos 60°)i + (10\sin 60°)j) \bullet ((\cos 60°)i + (\sin 60°)j)$$
$$= (10\cos 60°)(\cos 60°) + (10\sin 60°)(\sin 60°)$$
$$= 10((\cos 60°)^2 + (\sin 60°)^2) = 10$$

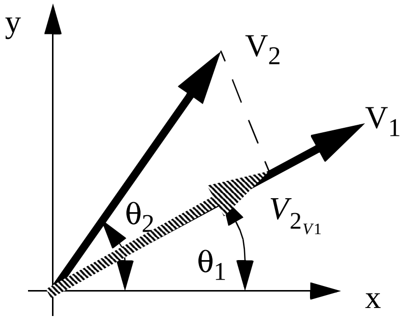
Correct

Now consider the case where we find the component of F in the x' direction. Again, this can be done using the dot product to project F onto a unit vector.

$$u_{x'} = \cos 45° i + \sin 45° j$$

$$F_{x'} = F \bullet \lambda_{x'} = ((10\cos 60°)i + (10\sin 60°)j) \bullet ((\cos 45°)i + (\sin 45°)j)$$
$$= (10\cos 60°)(\cos 45°) + (10\sin 60°)(\sin 45°)$$
$$= 10(\cos 60° \cos 45° + \sin 60° \sin 45°) = 10(\cos(60° - 45°))$$

Here we see a few cases where the dot product has been applied to find the vector projected onto a unit vector. Now finally consider the more general case,



First, by inspection, we can see that the component of V_2 (projected) in the direction of V_1 will be,

$$|V_{2_{V1}}| = |V_2|\cos(\theta_2 - \theta_1)$$

Next, we can manipulate this expression into the dot product form,

$$= |V_2|(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2)$$

$$= |V_2|[(\cos\theta_1 \underset{-}{i} + \sin\theta_1 \underset{-}{j}) \bullet (\cos\theta_2 \underset{-}{i} + \sin\theta_2 \underset{-}{j})]$$

$$= |V_2|\left[\frac{V_1}{|V_1|} \bullet \frac{V_2}{|V_2|}\right] = |V_2|\left[\frac{V_1 \bullet V_2}{|V_1||V_2|}\right] = \frac{V_1 \bullet V_2}{|V_1|} = V_2 \bullet \lambda_{V_1}$$

Or more generally,

$$|V_{2_{V1}}| = |V_2|\cos(\theta_2 - \theta_1) = |V_2|\left[\frac{V_1 \bullet V_2}{|V_1||V_2|}\right]$$

$$\therefore |V_2|\cos(\theta_2 - \theta_1) = |V_2|\left[\frac{V_1 \bullet V_2}{|V_1||V_2|}\right]$$

$$\therefore \cos(\theta_2 - \theta_1) = \left[\frac{V_1 \bullet V_2}{|V_1||V_2|}\right]$$

*Note that the dot product also works in 3D, and similar proofs are used.

34.5.3 Cross Product

• First, consider an example,

$$F = (-6.43\underline{i} + 7.66\underline{j} + 0\underline{k})N$$

$$d = (2\underline{i} + 0\underline{j} + 0\underline{k})m$$

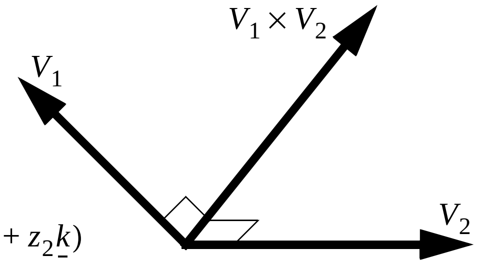
$$M = d \times F = \begin{bmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2m & 0m & 0m \\ -6.43N & 7.66N & 0N \end{bmatrix}$$

NOTE: note that the cross product here is for the right hand rule coordinates. If the left handed coordinate system is used F and d should be reversed.

$$\therefore M = (0m0N - 0m(7.66N))\underline{i} - (2m0N - 0m(-6.43N))\underline{j} + (2m(7.66N) - 0m(-6.43N))\underline{k} = 15.3\underline{k}(\text{mN})$$

NOTE: there are two things to note about the solution. First, it is a vector. Here there is only a z component because this vector points out of the page, and a rotation about this vector would rotate on the plane of the page. Second, this result is positive, because the positive sense is defined by the vector system. In this right handed system find the positive rotation by pointing your right hand thumb towards the positive axis (the ‘k’ means that the vector is about the z-axis here), and curl your fingers, that is the positive direction.

• The basic properties of the cross product are, The cross (or vector) product of two vectors will yield a new vector perpendicular to both vectors, with a magnitude that is a product of the two magnitudes.



$$V_1 \times V_2 = (x_1\underline{i} + y_1\underline{j} + z_1\underline{k}) \times (x_2\underline{i} + y_2\underline{j} + z_2\underline{k})$$

$$V_1 \times V_2 = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}$$

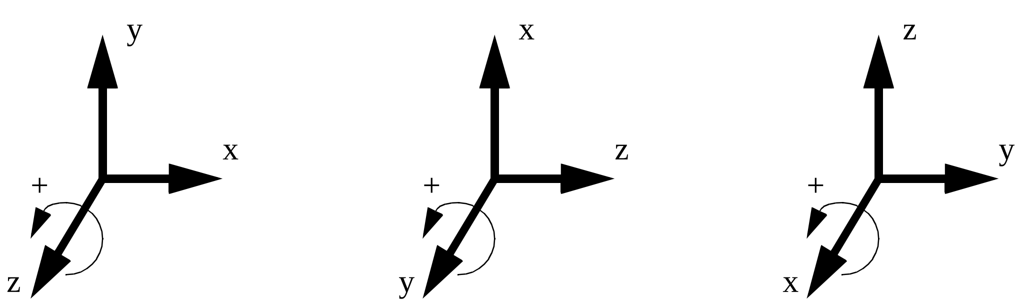
$$V_1 \times V_2 = (y_1z_2 - z_1y_2)\underline{i} + (z_1x_2 - x_1z_2)\underline{j} + (x_1y_2 - y_1x_2)\underline{k}$$

We can also find a unit vector normal ‘n’ to the vectors ‘V1’ and ‘V2’ using a cross product, divided by the magnitude.

$$\lambda_n = \frac{V_1 \times V_2}{|V_1 \times V_2|}$$

• When using a left/right handed coordinate system,

The positive orientation of angles and moments about an axis can be determined by pointing the thumb of the right hand along the axis of rotation. The fingers curl in the positive direction.



• The properties of the cross products are, The cross product is distributive, but not associative. This allows us to collect terms in a cross product operation, but we cannot change the order of the cross product.

$r_1 \times F + r_2 \times F = (r_1 + r_2) \times F$	DISTRIBUTIVE
$r \times F \neq F \times r$ but $r \times F = -(F \times r)$	NOT ASSOCIATIVE

34.5.4 Triple Product

When we want to do a cross product, followed by a dot product (called the mixed triple product), we can do both steps in one operation by finding the determinant of the following. An example of a problem that would use this shortcut is when a moment is found about one point on a pipe, and then the moment component twisting the pipe is found using the dot product.

$$(d \times F) \bullet u = \begin{vmatrix} u_x & u_y & u_z \\ d_x & d_y & d_z \\ F_x & F_y & F_z \end{vmatrix}$$

34.5.5 Matrices

• Matrices allow simple equations that drive a large number of repetitive calculations - as a result they are found in many computer applications.

• A matrix has the form seen below,

n columns

$$\begin{bmatrix} a_{11} & a_{21} & \dots & a_{n1} \\ a_{12} & a_{22} & \dots & a_{n2} \\ \dots & \dots & \dots & \dots \\ a_{1m} & a_{2m} & \dots & a_{nm} \end{bmatrix}$$

m rows

If n=m then the matrix is said to be square. Many applications require square matrices. We may also represent a matrix as a 1-by-3 for a vector.

• Matrix operations are available for many of the basic algebraic expressions, examples are given below. There are also many restrictions - many of these are indicated.

$$A = 2 \qquad B = \begin{bmatrix} 3 & 4 & 5 \\ 6 & 7 & 8 \\ 9 & 10 & 11 \end{bmatrix} \qquad C = \begin{bmatrix} 12 & 13 & 14 \\ 15 & 16 & 17 \\ 18 & 19 & 20 \end{bmatrix} \qquad D = \begin{bmatrix} 21 \\ 22 \\ 23 \end{bmatrix} \qquad E = \begin{bmatrix} 24 & 25 & 26 \end{bmatrix}$$

Addition/Subtraction

$$A + B = \begin{bmatrix} 3+2 & 4+2 & 5+2 \\ 6+2 & 7+2 & 8+2 \\ 9+2 & 10+2 & 11+2 \end{bmatrix} \qquad B + C = \begin{bmatrix} 3+12 & 4+13 & 5+14 \\ 6+15 & 7+16 & 8+17 \\ 9+18 & 10+19 & 11+20 \end{bmatrix}$$

$$B + D = \text{not valid}$$

$$B - A = \begin{bmatrix} 3-2 & 4-2 & 5-2 \\ 6-2 & 7-2 & 8-2 \\ 9-2 & 10-2 & 11-2 \end{bmatrix} \qquad B + C = \begin{bmatrix} 3-12 & 4-13 & 5-14 \\ 6-15 & 7-16 & 8-17 \\ 9-18 & 10-19 & 11-20 \end{bmatrix}$$

$$B - D = \text{not valid}$$

Multiplication/Division

$$A \cdot B = \begin{bmatrix} 3(2) & 4(2) & 5(2) \\ 6(2) & 7(2) & 8(2) \\ 9(2) & 10(2) & 11(2) \end{bmatrix} \qquad \frac{B}{A} = \begin{bmatrix} \frac{3}{2} & \frac{4}{2} & \frac{5}{2} \\ \frac{6}{2} & \frac{7}{2} & \frac{8}{2} \\ \frac{9}{2} & \frac{10}{2} & \frac{11}{2} \end{bmatrix}$$

$$B \cdot D = \begin{bmatrix} (3 \cdot 21 + 4 \cdot 22 + 5 \cdot 23) \\ (6 \cdot 21 + 7 \cdot 22 + 8 \cdot 23) \\ (9 \cdot 21 + 10 \cdot 22 + 11 \cdot 23) \end{bmatrix} \qquad D \cdot E = 21 \cdot 24 + 22 \cdot 25 + 23 \cdot 26$$

$$B \cdot C = \begin{bmatrix} (3 \cdot 12 + 4 \cdot 15 + 5 \cdot 18) & (3 \cdot 13 + 4 \cdot 16 + 5 \cdot 19) & (3 \cdot 14 + 4 \cdot 17 + 5 \cdot 20) \\ (6 \cdot 12 + 7 \cdot 15 + 8 \cdot 18) & (6 \cdot 13 + 7 \cdot 16 + 8 \cdot 19) & (6 \cdot 14 + 7 \cdot 17 + 8 \cdot 20) \\ (9 \cdot 12 + 10 \cdot 15 + 11 \cdot 18) & (9 \cdot 13 + 10 \cdot 16 + 11 \cdot 19) & (9 \cdot 14 + 10 \cdot 17 + 11 \cdot 20) \end{bmatrix}$$

$\frac{B}{C}, \frac{B}{D}, \frac{D}{B}$, etc = not allowed (see inverse)

Note: To multiply matrices, the first matrix must have the same number of columns as the second matrix has rows.

Determinant

$$|B| = 3 \cdot \begin{vmatrix} 7 & 8 \\ 10 & 11 \end{vmatrix} - 4 \cdot \begin{vmatrix} 6 & 8 \\ 9 & 11 \end{vmatrix} + 5 \cdot \begin{vmatrix} 6 & 7 \\ 9 & 10 \end{vmatrix} = 3 \cdot (-3) - 4 \cdot (-6) + 5 \cdot (-3) = 0$$

$$\begin{vmatrix} 7 & 8 \\ 10 & 11 \end{vmatrix} = (7 \cdot 11) - (8 \cdot 10) = -3$$

$$\begin{vmatrix} 6 & 8 \\ 9 & 11 \end{vmatrix} = (6 \cdot 11) - (8 \cdot 9) = -6$$

$$\begin{vmatrix} 6 & 7 \\ 9 & 10 \end{vmatrix} = (6 \cdot 10) - (7 \cdot 9) = -3$$

$|D|, |E| =$ not valid (matrices not square)

Transpose

$$B^T = \begin{bmatrix} 3 & 6 & 9 \\ 4 & 7 & 10 \\ 5 & 8 & 11 \end{bmatrix} \qquad D^T = \begin{bmatrix} 21 & 22 & 23 \end{bmatrix} \qquad E^T = \begin{bmatrix} 24 \\ 25 \\ 26 \end{bmatrix}$$

Adjoint

$$||B|| = \begin{bmatrix} \begin{vmatrix} 7 & 8 \\ 10 & 11 \end{vmatrix} & -\begin{vmatrix} 6 & 8 \\ 10 & 11 \end{vmatrix} & \begin{vmatrix} 6 & 7 \\ 9 & 10 \end{vmatrix} \\ -\begin{vmatrix} 4 & 5 \\ 10 & 11 \end{vmatrix} & \begin{vmatrix} 3 & 5 \\ 9 & 11 \end{vmatrix} & -\begin{vmatrix} 3 & 4 \\ 9 & 10 \end{vmatrix} \\ \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} & -\begin{vmatrix} 3 & 5 \\ 6 & 8 \end{vmatrix} & \begin{vmatrix} 3 & 4 \\ 6 & 7 \end{vmatrix} \end{bmatrix}^T$$

The matrix of determinant to the left is made up by getting rid of the row and column of the element, and then finding the determinant of what is left. Note the sign changes on alternating elements.

$$||D|| = \text{invalid (must be square)}$$

Inverse

$$D = B \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

To solve this equation for x,y,z we need to move B to the left hand side. To do this we use the inverse.

$$B^{-1}D = B^{-1} \cdot B \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$B^{-1}D = I \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$B^{-1} = \frac{||B||}{|B|}$$

In this case B is singular, so the inverse is undetermined, and the matrix is indeterminate.

$$D^{-1} = \text{invalid (must be square)}$$

Identity Matrix

This is a square matrix that is the matrix equivalent to ‘1’.

$$B \cdot I = I \cdot B = B$$

$$D \cdot I = I \cdot D = D$$

$$B^{-1} \cdot B = I$$

$$\begin{bmatrix} 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{etc=I}$$

• The eigenvalue of a matrix is found using,

$$|A - \lambda I| = 0$$

34.5.6 Solving Linear Equations with Matrices

• We can solve systems of equations using the inverse matrix,

Given,

$$2 \cdot x + 4 \cdot y + 3 \cdot z = 5$$

$$9 \cdot x + 6 \cdot y + 8 \cdot z = 7$$

$$11 \cdot x + 13 \cdot y + 10 \cdot z = 12$$

Write down the matrix, then rearrange, and solve.

$$\begin{bmatrix} 2 & 4 & 3 \\ 9 & 6 & 8 \\ 11 & 13 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 12 \end{bmatrix} \qquad \therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 & 4 & 3 \\ 9 & 6 & 8 \\ 11 & 13 & 10 \end{bmatrix}^{-1} \begin{bmatrix} 5 \\ 7 \\ 12 \end{bmatrix} = \begin{bmatrix} \\ \\ \end{bmatrix}$$

• We can solve systems of equations using Cramer's rule (with determinants),

Given,

$$2 \cdot x + 4 \cdot y + 3 \cdot z = 5$$

$$9 \cdot x + 6 \cdot y + 8 \cdot z = 7$$

$$11 \cdot x + 13 \cdot y + 10 \cdot z = 12$$

Write down the coefficient and parameter matrices,

$$A = \begin{bmatrix} 2 & 4 & 3 \\ 9 & 6 & 8 \\ 11 & 13 & 10 \end{bmatrix} \qquad B = \begin{bmatrix} 5 \\ 7 \\ 12 \end{bmatrix}$$

Calculate the determinant for A (this will be reused), and calculate the determinants for matrices below. Note: when trying to find the first parameter 'x' we replace the first column in A with B.

$$x = \frac{\begin{vmatrix} 5 & 4 & 3 \\ 7 & 6 & 8 \\ 12 & 13 & 10 \end{vmatrix}}{|A|} =$$

$$y = \frac{\begin{vmatrix} 2 & 5 & 3 \\ 9 & 7 & 8 \\ 11 & 12 & 10 \end{vmatrix}}{|A|} =$$

$$z = \frac{\begin{vmatrix} 2 & 4 & 5 \\ 9 & 6 & 7 \\ 11 & 13 & 12 \end{vmatrix}}{|A|} =$$

34.5.7 Practice Problems

1. Perform the matrix operations below.

Multiply

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 10 \\ 11 \\ 12 \end{bmatrix} =$$

Determinant

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 2 & 6 \\ 7 & 8 & 9 \end{vmatrix} =$$

Inverse

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 2 & 6 \\ 7 & 8 & 9 \end{bmatrix}^{-1} =$$

ANS.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 10 \\ 11 \\ 12 \end{bmatrix} = \begin{bmatrix} 68 \\ 167 \\ 266 \end{bmatrix}$$

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 2 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 36$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 2 & 6 \\ 7 & 8 & 9 \end{bmatrix}^{-1} = \begin{bmatrix} -0.833 & 0.167 & 0.167 \\ 0.167 & -0.333 & 0.167 \\ 0.5 & 0.167 & -0.167 \end{bmatrix}$$

2. Perform the vector operations below,

$$A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad B = \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix}$$

Cross Product $A \times B =$

Dot Product $A \bullet B =$

ANS.

$A \times B = (-4, 17, -10)$

$A \bullet B = 13$

4. Solve the following equations using any technique,

$$5x - 2y + 4z = -1$$
$$6x + 7y + 5z = -2$$
$$2x - 3y + 6z = -3$$

ANS.

x= 0.273
y= -0.072
z= -0.627

5. Solve the following set of equations using a) Cramer's rule and b) an inverse matrix.

$$2x + 3y = 4$$

$$5x + 1y = 0$$

(ans. $\begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$

a) $$x = \frac{\left|\begin{bmatrix} 4 & 3 \\ 0 & 1 \end{bmatrix}\right|}{\left|\begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix}\right|} = \frac{4}{-13} \qquad y = \frac{\left|\begin{bmatrix} 2 & 4 \\ 5 & 0 \end{bmatrix}\right|}{\left|\begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix}\right|} = \frac{-20}{-13} = \frac{20}{13}$$

b) $$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 4 \\ 0 \end{bmatrix} = \frac{\begin{bmatrix} 1 & -(5) \\ -(3) & 2 \end{bmatrix}^T}{\left|\begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix}\right|}\begin{bmatrix} 4 \\ 0 \end{bmatrix} = \frac{\begin{bmatrix} 1 & -3 \\ -5 & 2 \end{bmatrix}}{-13}\begin{bmatrix} 4 \\ 0 \end{bmatrix} = \left(-\frac{1}{13}\right)\begin{bmatrix} 4 \\ -20 \end{bmatrix} = \begin{bmatrix} -\frac{4}{13} \\ \frac{20}{13} \end{bmatrix}$$

6. Perform the following matrix calculation. Show all work.

$$\left[\begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix}\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} + \begin{bmatrix} L \\ M \\ N \end{bmatrix}\right]^T$$

(ans.

$$\left[\begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix}\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} + \begin{bmatrix} L \\ M \\ N \end{bmatrix}\right]^T = \left[\begin{bmatrix} AX + BY + CZ \\ DX + EY + FZ \\ GX + YH + IZ \end{bmatrix} + \begin{bmatrix} L \\ M \\ N \end{bmatrix}\right]^T = \begin{bmatrix} AX + BY + CZ + L \\ DX + EY + FZ + M \\ GX + YH + IZ + N \end{bmatrix}^T$$
$$= \begin{bmatrix} AX + BY + CZ + L & DX + EY + FZ + M & GX + YH + IZ + N \end{bmatrix}$$

7. Perform the matrix calculations given below.

a) $$\begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix}\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} =$$

b) $$\begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix}\begin{bmatrix} X & Y & Z \end{bmatrix} =$$

8. Find the dot product, and the cross product, of the vectors A and B below.

$$A = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad B = \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

(ans.

$$A \bullet B = xp + yq + zr$$

$$A \times B = \begin{vmatrix} i & j & k \\ x & y & z \\ p & q & r \end{vmatrix} = i(yr - zq) + j(zp - xr) + k(xq - yp) = \begin{bmatrix} yr - zq \\ zp - xr \\ xq - yp \end{bmatrix}$$

9. Perform the following matrix calculations.

a) $\begin{bmatrix} a \\ b \\ c \end{bmatrix}^T \begin{bmatrix} d & e & f \\ g & h & k \\ m & n & p \end{bmatrix}$ b) $\left| \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right|$ c) $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1}$

(ans.

a) $$\begin{bmatrix} a \\ b \\ c \end{bmatrix}^T \begin{bmatrix} d & e & f \\ g & h & k \\ m & n & p \end{bmatrix} = \begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} d & e & f \\ g & h & k \\ m & n & p \end{bmatrix} = \begin{bmatrix} (ad + bg + cm) & (ae + bh + cn) & (af + bk + cp) \end{bmatrix}$$

b) $$\left| \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right| = ad - bc$$

c) $$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{adj}}{\left| \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right|} = \frac{\begin{bmatrix} d & -c \\ -b & a \end{bmatrix}}{ad - bc} = \begin{bmatrix} \frac{d}{ad - bc} & \frac{-c}{ad - bc} \\ \frac{-b}{ad - bc} & \frac{a}{ad - bc} \end{bmatrix}$$

10. Find the value of ‘x’ for the following system of equations.

$$x + 2y + 3z = 5$$

$$x + 4y + 8z = 0$$

$$4x + 2y + z = 1$$

(ans.
$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 8 \\ 4 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}$$

$$x = \frac{\begin{bmatrix} 5 & 2 & 3 \\ 0 & 4 & 8 \\ 1 & 2 & 1 \end{bmatrix}}{\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 8 \\ 4 & 2 & 1 \end{bmatrix}} = \frac{5(4-16) + 2(8-0) + 3(0-4)}{1(4-16) + 2(32-1) + 3(2-16)} = \frac{-60 + 16 - 12}{-12 + 62 - 42} = \frac{-56}{8} = -7$$

11. Perform the matrix calculations given below.

a) $$\begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} =$$

b) $$\begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix} \begin{bmatrix} X & Y & Z \end{bmatrix} =$$

c) $$det \begin{bmatrix} A & B \\ C & D \end{bmatrix} =$$

d) $$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^A =$$

e) $$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} =$$

12. Solve the following set of equations with the specified methods.

$$3x + 5y = 7$$
$$4x - 6y = 2$$

a) Inverse matrix

b) Cramer's rule

c) Gauss-Jordan row reduction

d) Substitution

34.6 CALCULUS

• NOTE: Calculus is very useful when looking at real systems. Many students are turned off by the topic because they "don't get it". But, the secret to calculus is to remember that there is no single "truth" - it is more a loose collection of tricks and techniques. Each one has to be learned separately, and when needed you must remember it, or know where to look.

34.6.1 Single Variable Functions

34.6.1.1 - Differentiation

• The basic principles of differentiation are, Both u, v and w are functions of x, but this is not shown for brevity.

Also note that C is used as a constant, and all angles are in radians.

$$\frac{d}{dx}(C) = 0$$

$$\frac{d}{dx}(Cu) = (C)\frac{d}{dx}(u)$$

$$\frac{d}{dx}(u + v + \dots) = \frac{d}{dx}(u) + \frac{d}{dx}(v) + \dots$$

$$\frac{d}{dx}(u^n) = (nu^{n-1})\frac{d}{dx}(u)$$

$$\frac{d}{dx}(uv) = (u)\frac{d}{dx}(v) + (v)\frac{d}{dx}(u)$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \left(\frac{v}{v^2}\right)\frac{d}{dx}(u) - \left(\frac{u}{v^2}\right)\frac{d}{dx}(v)$$

$$\frac{d}{dx}(uvw) = (uv)\frac{d}{dx}(w) + (uw)\frac{d}{dx}(v) + (vw)\frac{d}{dx}(u)$$

$$\frac{d}{dx}(y) = \frac{d}{du}(y)\frac{d}{dx}(u) = \text{chain rule}$$

$$\frac{d}{dx}(u) = \frac{1}{\frac{d}{du}(x)}$$

$$\frac{d}{dx}(y) = \frac{\frac{d}{du}(y)}{\frac{d}{du}(x)}$$

- Differentiation rules specific to basic trigonometry and logarithm functions

$$\frac{d}{dx}(\sin u) = (\cos u)\frac{d}{dx}(u)$$

$$\frac{d}{dx}(\cos u) = (-\sin u)\frac{d}{dx}(u)$$

$$\frac{d}{dx}(\tan u) = \left(\frac{1}{\cos u}\right)^2\frac{d}{dx}(u)$$

$$\frac{d}{dx}(e^u) = (e^u)\frac{d}{dx}(u)$$

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

$$\frac{d}{dx}(\cot u) = (-\csc u)^2\frac{d}{dx}(u)$$

$$\frac{d}{dx}(\sec u) = (\tan u \sec u)\frac{d}{dx}(u)$$

$$\frac{d}{dx}(\csc u) = (-\csc u \cot u)\frac{d}{dx}(u)$$

$$\frac{d}{dx}(\sinh u) = (\cosh u)\frac{d}{dx}(u)$$

$$\frac{d}{dx}(\cosh u) = (\sinh u)\frac{d}{dx}(u)$$

$$\frac{d}{dx}(\tanh u) = (\operatorname{sech} u)^2\frac{d}{dx}(u)$$

• L'Hospital's rule can be used when evaluating limits that go to infinity.

$$\lim_{x \to a}\left(\frac{f(x)}{g(x)}\right) = \lim_{x \to a}\left(\frac{\left(\frac{d}{dt}\right)f(x)}{\left(\frac{d}{dt}\right)g(x)}\right) = \lim_{x \to a}\left(\frac{\left(\frac{d}{dt}\right)^2 f(x)}{\left(\frac{d}{dt}\right)^2 g(x)}\right) = \ldots$$

• Some techniques used for finding derivatives are, Leibnitz’s Rule, (notice the form is similar to the binomial equation) can be used for finding the derivatives of multiplied functions.

$$\left(\frac{d}{dx}\right)^n(uv) = \left(\frac{d}{dx}\right)^0(u)\left(\frac{d}{dx}\right)^n(v) + \binom{n}{1}\left(\frac{d}{dx}\right)^1(u)\left(\frac{d}{dx}\right)^{n-1}(v)$$

$$\binom{n}{2}\left(\frac{d}{dx}\right)^2(u)\left(\frac{d}{dx}\right)^{n-2}(v) + \ldots + \binom{n}{n}\left(\frac{d}{dx}\right)^n(u)\left(\frac{d}{dx}\right)^0(v)$$

34.6.1.2 - Integration

• Some basic properties of integrals include,

In the following expressions, u, v, and w are functions of x. in addition to this, C is a constant. and all angles are radians.

$$\int C dx = ax + C$$

$$\int Cf(x)dx = C\int f(x)dx$$

$$\int (u + v + w + \ldots)dx = \int u dx + \int v dx + \int w dx + \ldots$$

$$\int u dv = uv - \int v du = \text{ integration by parts}$$

$$\int f(Cx)dx = \frac{1}{C}\int f(u)du \qquad u = Cx$$

$$\int F(f(x))dx = \int F(u)\frac{d}{du}(x)du = \int \frac{F(u)}{f'(x)}du \qquad u = f(x)$$

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \qquad \int \frac{1}{x}dx = \ln|x| + C$$

$$\int a^x dx = \frac{a^x}{\ln a} + C \qquad \int e^x dx = e^x + C$$

• Some of the trigonometric integrals are,

$$\int \sin x dx = -\cos x + C$$

$$\int \cos x dx = \sin x + C$$

$$\int (\sin x)^2 dx = -\frac{\sin x \cos x + x}{2} + C$$

$$\int (\cos x)^2 dx = \frac{\sin x \cos x + x}{2} + C$$

$$\int (\sin x)^3 dx = -\frac{\cos x((\sin x)^2 + 2)}{3} + C$$

$$\int (\cos x)^3 dx = \frac{\sin x((\cos x)^2 + 2)}{3} + C$$

$$\int (\cos x)^4 dx = \frac{3x}{8} + \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + C$$

$$\int \cos x (\sin x)^n dx = \frac{(\sin x)^{n+1}}{n+1} + C$$

$$\int \sinh x dx = \cosh x + C$$

$$\int \cosh x dx = \sinh x + C$$

$$\int \tanh x dx = \ln(\cosh x) + C$$

$$\int x \cos(ax) dx = \frac{\cos(ax)}{a^2} + \frac{x}{a}\sin(ax) + C$$

$$\int x^2 \cos(ax) dx = \frac{2x\cos(ax)}{a^2} + \frac{a^2x^2 - 2}{a^3}\sin(ax) + C$$

• Some other integrals of use that are basically functions of x are,

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C$$

$$\int (a+bx)^{-1} dx = \frac{\ln|a+bx|}{b} + C$$

$$\int (a+bx^2)^{-1} dx = \frac{1}{2\sqrt{(-b)a}} \ln\left(\frac{\sqrt{a} + 2\sqrt{-b}}{\sqrt{a} - x\sqrt{-b}}\right) + C,\ a > 0,\ b < 0$$

$$\int x(a+bx^2)^{-1} dx = \frac{\ln(bx^2+a)}{2b} + C$$

$$\int x^2(a+bx^2)^{-1} dx = \frac{x}{b} - \frac{a}{b\sqrt{ab}} \operatorname{atan}\left(\frac{x\sqrt{ab}}{a}\right) + C$$

$$\int (a^2 - x^2)^{-1} dx = \frac{1}{2a} \ln\left(\frac{a+x}{a-x}\right) + C,\ a^2 > x^2$$

$$\int (a+bx)^{-1} dx = \frac{2\sqrt{a+bx}}{b} + C$$

CORRECT??

$$\int x(x^2 \pm a^2)^{-\frac{1}{2}} dx = \sqrt{x^2 \pm a^2} + C$$

$$\int (a+bx+cx^2)^{-1} dx = \frac{1}{\sqrt{c}} \ln\left[\sqrt{a+bx+cx^2} + x\sqrt{c} + \frac{b}{2\sqrt{c}}\right] + C,\ c > 0$$

$$\int (a+bx+cx^2)^{-1} dx = \frac{1}{\sqrt{-c}} \operatorname{asin}\left[\frac{-2cx - b}{\sqrt{b^2 - 4ac}}\right] + C,\ c < 0$$

$$\int (a+bx)^{\frac{1}{2}}dx = \frac{2}{3b}(a+bx)^{\frac{3}{2}}$$

$$\int (a+bx)^{\frac{1}{2}}dx = \frac{2}{3b}(a+bx)^{\frac{3}{2}}$$

$$\int x(a+bx)^{\frac{1}{2}}dx = -\frac{2(2a-3bx)(a+bx)^{\frac{3}{2}}}{15b^2}$$

$$\int (1+a^2x^2)^{\frac{1}{2}}dx = \frac{x(1+a^2x^2)^{\frac{1}{2}} + \frac{\ln\left(x+\left(\frac{1}{a^2}+x^2\right)^{\frac{1}{2}}\right)}{a}}{2}$$

$$\int x(1+a^2x^2)^{\frac{1}{2}}dx = \frac{a\left(\frac{1}{a^2}+x^2\right)^{\frac{3}{2}}}{3}$$

$$\int x^2(1+a^2x^2)^{\frac{1}{2}}dx = \frac{ax}{4}\left(\frac{1}{a^2}+x^2\right)^{\frac{3}{2}} - \frac{8}{8a^2}x(1+a^2x^2)^{\frac{1}{2}} - \frac{\ln\left(x+\left(\frac{1}{a^2}+x^2\right)^{\frac{1}{2}}\right)}{8a^3}$$

$$\int (1-a^2x^2)^{\frac{1}{2}}dx = \frac{1}{2}\left[x(1-a^2x^2)^{\frac{1}{2}} + \frac{\operatorname{asin}(ax)}{a}\right]$$

$$\int x(1-a^2x^2)^{\frac{1}{2}}dx = -\frac{a}{3}\left(\frac{1}{a^2}-x^2\right)^{\frac{3}{2}}$$

$$\int x^2(a^2-x^2)^{\frac{1}{2}}dx = -\frac{x}{4}(a^2-x^2)^{\frac{3}{2}} + \frac{1}{8}\left[x(a^2-x^2)^{\frac{1}{2}} + a^2\operatorname{asin}\left(\frac{x}{a}\right)\right]$$

$$\int (1+a^2x^2)^{-\frac{1}{2}}dx = \frac{1}{a}\ln\left[x+\left(\frac{1}{a^2}+x^2\right)^{\frac{1}{2}}\right]$$

$$\int (1-a^2x^2)^{-\frac{1}{2}}dx = \frac{1}{a}\operatorname{asin}(ax) = -\frac{1}{a}\operatorname{acos}(ax)$$

• Integrals using the natural logarithm base ‘e’,

$$\int e^{ax} dx = \frac{e^{ax}}{a} + C$$

$$\int x e^{ax} dx = \frac{e^{ax}}{a^2}(ax - 1) + C$$

34.6.2 Vector Calculus

• When dealing with large and/or time varying objects or phenomenon we must be able to describe the state at locations, and as a whole. To do this vectors are a very useful tool.

• Consider a basic function and how it may be represented with partial derivatives.

$$y = f(x, y, z)$$

We can write this in differential form, but the right hand side must contain partial derivatives. If we separate the operators from the function, we get a simpler form. We can then look at them as the result of a dot product, and divide it into two vectors.

$$(d)y = \left(\left(\frac{\partial}{\partial x}\right) f(x, y, z)\right) dx + \left(\left(\frac{\partial}{\partial y}\right) f(x, y, z)\right) dy + \left(\left(\frac{\partial}{\partial z}\right) f(x, y, z)\right) dz$$

$$(d)y = \left[\left(\frac{\partial}{\partial x}\right) dx + \left(\frac{\partial}{\partial y}\right) dy + \left(\frac{\partial}{\partial z}\right) dz\right] f(x, y, z)$$

$$(d)y = \left[\left(\frac{\partial}{\partial x}\underline{i} + \frac{\partial}{\partial y}\underline{j} + \frac{\partial}{\partial z}\underline{k}\right) \bullet (dx\underline{i} + dy\underline{j} + dz\underline{k})\right] f(x, y, z)$$

We then replace these vectors with the operators below. In this form we can manipulate the equation easily (whereas the previous form was very awkward).

$$(d)y = [\nabla \bullet dX] f(x, y, z)$$

$$(d)y = \nabla f(x, y, z) \bullet dX$$

$$(d)y = |\nabla f(x, y, z)||dX| \cos\theta$$

In summary,

$$\nabla = \frac{\partial}{\partial x}\underline{i} + \frac{\partial}{\partial y}\underline{j} + \frac{\partial}{\partial z}\underline{k}$$

$$F = F_x\underline{i} + F_y\underline{j} + F_z\underline{k}$$

$\nabla \bullet F$ = the divergence of function F

$\nabla \times F$ = the curl of function F

• Gauss's or Green's or divergence theorem is given below. Both sides give the flux across a surface, or out of a volume. This is very useful for dealing with magnetic fields.

$$\int_V (\nabla \bullet F) dV = \oint_A F dA$$

where,

V, A = a volume V enclosed by a surface area A

F = a field or vector value over a volume

• Stoke’s theorem is given below. Both sides give the flux across a surface, or out of a volume. This is very useful for dealing with magnetic fields.

$$\int_A (\nabla \times F) dA = \oint_L F dL$$

where,

A, L = A surface area A, with a bounding parimeter of length L

F = a field or vector value over a volume

34.6.3 Differential Equations

• Solving differential equations is not very challenging, but there are a number of forms that need to be remembered.

• Another complication that often occurs is that the solution of the equations may vary depending upon boundary or initial conditions. An example of this is a mass spring combination. If they are initially at rest then they will stay at rest, but if there is some disturbance, then they will oscillate indefinitely.

• We can judge the order of these equations by the highest order derivative in the equation.

• Note: These equations are typically shown with derivatives only, when integrals occur they are typically eliminated by taking derivatives of the entire equation.

• Some of the terms used when describing differential equations are,

ordinary differential equations - if all the derivatives are of a single variable. In the example below ’x’ is the variable with derivatives.

e.g., $$\left(\frac{d}{dt}\right)^2 x + \left(\frac{d}{dt}\right) x = y$$

first-order differential equations - have only first-order derivatives,

e.g., $$\left(\frac{d}{dt}\right) x + \left(\frac{d}{dt}\right) y = 2$$

second-order differential equations - have at least on second derivative,

e.g., $$\left(\frac{d}{dt}\right)^2 x + \left(\frac{d}{dt}\right)y = 2$$

higher order differential equations - have at least one derivative that is higher than second-order.
partial differential equations - these equations have partial derivatives

• Note: when solving these equations it is common to hit blocks. In these cases backtrack and try another approach.

• linearity of a differential equation is determined by looking at the dependant variables in the equation. The equation is linear if they appear with an exponent other than 1.

eg.

$y'' + y' + 2 = 5x$	linear
$(y'')^2 + y' + 2 = 5x$	non-linear
$y'' + (y')^3 + 2 = 5x$	non-linear
$y'' + \sin(y') + 2 = 5x$	non-linear

34.6.3.1 - First-order Differential Equations

• These systems tend to have a relaxed or passive nature in real applications.

• Examples of these equations are given below,

$$y' + 2xy^2 - 4x^3 = 0$$
$$y' - 2y = 0$$

• Typical methods for solving these equations include,

guessing then testing
separation

homogeneous

34.6.3.1.1 - Guessing

• In this technique we guess at a function that will satisfy the equation, and test it to see if it works.

$y' + y = 0$ the given equation

$y = Ce^{-t}$ the guess

now try to substitute into the equation

$y' = -Ce^{-t}$

$y' + y = -Ce^{-t} + Ce^{-t} = 0$ therefore the guess worked - it is correct

$$\boxed{y = Ce^{-t}}$$

• The previous example showed a general solution (i.e., the value of ’C’ was not found). We can also find a particular solution.

$y = Ce^{-t}$ a general solution

$y = 5e^{-t}$ a particular solution

34.6.3.1.2 - Separable Equations

• In a separable equation the differential can be split so that it is on both sides of the equation. We then integrate to get the solution. This typically means there is only a single derivative term.

e.g., $$\frac{dx}{dy} + y^2 + 2y + 3 = 0$$

$$\therefore dx = (-y^2 - 2y - 3)dy$$

$$\therefore x = \frac{-y^3}{3} - y^2 - 3y + C$$

e.g., $$\frac{dx}{dy} + x = 0$$

$$\therefore \left(-\frac{1}{x}\right)dx = dy$$

$$\therefore \ln(-x) = y$$

34.6.3.1.3 - Homogeneous Equations and Substitution

• These techniques depend upon finding some combination of the variables in the equation that can be replaced with another variable to simplify the equation. This technique requires a bit of guessing about what to substitute for, and when it is to be applied.

e.g., $\frac{dy}{dx} = \frac{y}{x} - 1$ the equation given

$u = \frac{y}{x}$ the substitution chosen

Put the substitution in and solve the differential equation,

$$\frac{dy}{dx} = u - 1$$

$$\therefore u + x\frac{du}{dx} = u - 1$$

$$\therefore \frac{du}{dx} = \frac{-1}{x}$$

$$\therefore -\frac{du}{dx} = \frac{1}{x}$$

$$\therefore -u = \ln(x) + C$$

Substitute the results back into the original substitution equation to get rid of 'u',

$$-\frac{y}{x} = \ln(x) + C$$

$$\therefore y = -x\ln(x) - Cx$$

34.6.3.2 - Second-order Differential Equations

- These equations have at least one second-order derivative.

- In engineering we will encounter a number of forms,

- homogeneous
- nonhomogeneous

34.6.3.2.1 - Linear Homogeneous

- These equations will have a standard form,

$$\left(\frac{d}{dt}\right)^2 y + A\left(\frac{d}{dt}\right)y + By = 0$$

• An example of a solution is,

e.g.,
$$\left(\frac{d}{dt}\right)^2 y + 6\left(\frac{d}{dt}\right)y + 3y = 0$$

Guess,

$$y = e^{Bt}$$

$$\left(\frac{d}{dt}\right)y = Be^{Bt}$$

$$\left(\frac{d}{dt}\right)^2 y = B^2 e^{Bt}$$

substitute and solve for B,

$$B^2 e^{Bt} + 6Be^{Bt} + 3e^{Bt} = 0$$

$$B^2 + 6B + 3 = 0$$

$$B = -3 + 2.449j, -3 - 2.449j$$

substitute and solve for B,

$$y = e^{(-3 + 2.449j)t}$$

$$y = e^{-3t} e^{2.449jt}$$

$$y = e^{-3t}(\cos(2.449t) + j\sin(2.449t))$$

Note: if both the roots are the same,

$$y = C_1 e^{Bt} + C_2 t e^{Bt}$$

34.6.3.2.2 - Nonhomogeneous Linear Equations

• These equations have the general form,

$$\left(\frac{d}{dt}\right)^2 y + A\left(\frac{d}{dt}\right)y + By = Cx$$

• to solve these equations we need to find the homogeneous and particular solutions and then add the two solutions.

$$y = y_h + y_p$$

to find yh solve,

$$\left(\frac{d}{dt}\right)^2 y + A\left(\frac{d}{dt}\right)y + B = 0$$

to find yp guess at a value of y and then test for validity, A good table of guesses is,

Cx form	Guess
A	C
$Ax + B$	$Cx + D$
e^{Ax}	Ce^{Ax} Cxe^{Ax}
$B\sin(Ax)$ or $B\cos(Ax)$	$C\sin(Ax) + D\cos(Ax)$ or $Cx\sin(Ax) + xD\cos(Ax)$

• Consider the example below,

$$\left(\frac{d}{dt}\right)^2 y + \left(\frac{d}{dt}\right) y - 6y = e^{-2x}$$

First solve for the homogeneous part,

$$\left(\frac{d}{dt}\right)^2 y + \left(\frac{d}{dt}\right) y - 6y = 0 \qquad \text{try} \qquad y = e^{Bx}$$

$$\left(\frac{d}{dt}\right) y = Be^{Bx}$$

$$\left(\frac{d}{dt}\right)^2 y = B^2 e^{Bx}$$

$$B^2 e^{Bx} + Be^{Bx} - 6e^{Bx} = 0$$

$$B^2 + B - 6 = 0$$

$$B = -3, 2$$

$$y_h = e^{-3x} + e^{2x}$$

Next, solve for the particular part. We will guess the function below.

$$y = Ce^{-2x}$$

$$\left(\frac{d}{dt}\right) y = -2Ce^{-2x}$$

$$\left(\frac{d}{dt}\right)^2 y = 4Ce^{-2x}$$

$$4Ce^{-2x} + -2Ce^{-2x} - 6Ce^{-2x} = e^{-2x}$$

$$4C - 2C - 6C = 1$$

$$C = 0.25$$

$$y_p = 0.25e^{-2x}$$

Finally,

$$y = e^{-3x} + e^{2x} + 0.25e^{-2x}$$

34.6.3.3 - Higher Order Differential Equations

34.6.3.4 - Partial Differential Equations

• Partial difference equations become critical with many engineering applications involving flows, etc.

34.6.4 Other Calculus Stuff

• The Taylor series expansion can be used to find polynomial approximations of functions.

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \ldots + \frac{f^{(n-1)}(a)(x-a)^{n-1}}{(n-1)!}$$

34.6.5 Practice Problems

1. Find the derivative of the function below with respect to time.

$$\frac{3t}{(2+t)^2} + e^{2t}$$

(ans. $\left(\frac{d}{dt}\right)\left(\frac{3t}{(2+t)^2} + e^{2t}\right) = \left(\frac{d}{dt}\right)(3t(2+t)^{-2}) + \left(\frac{d}{dt}\right)(e^{2t}) = 3(2+t)^{-2} - 6t(2+t)^{-3} + 2e^{2t}$

2. Solve the following differential equation, given the initial conditions at t=0s.

$$x'' + 4x' + 4x = 5t \qquad x_0 = 0 \qquad x'_0 = 0$$

(ans. homogeneous solution:

$$x'' + 4x' + 4x = 0$$

guess: $x_h = e^{At}$ $\quad x_h' = Ae^{At}$ $\quad x_h'' = A^2e^{At}$

$$A^2e^{At} + 4Ae^{At} + 4e^{At} = 0 = A^2 + 4A + 4 = (A+2)(A+2)$$

$$x_h = C_1e^{-2t} + C_2te^{-2t}$$

particular solution:

guess: $x_p = At + B$ $\quad x_p' = A$ $\quad x_p'' = 0$

$$(0) + 4(A) + 4(At + B) = 5t$$

$$(4A + 4B) + (4A)t = (0) + (5)t$$

$$A = \frac{5}{4} = 1.25 \qquad 4(1.25) + 4B = 0 \qquad B = -1.25$$

$$x_p = 1.25t - 1.25$$

combine and solve for constants:

$$x(t) = x_h + x_p = C_1e^{-2t} + C_2te^{-2t} + 1.25t - 1.25$$

for x(0) = 0 $\quad 0 = C_1e^{-2(0)} + C_2(0)e^{-2(0)} + 1.25(0) - 1.25 = C_1 - 1.25$

$$C_1 = 1.25$$

for (d/dt)x(0) = 0 $\quad 0 = (-2)C_1e^{-2(0)} + (-2)C_2(0)e^{-2(0)} + C_2e^{-2(0)} + 1.25$

$$-2C_1 + C_2 + 1.25 = 0 = -2(1.25) + C_2 + 1.25 \qquad C_2 = 1.25$$

$$x(t) = 1.25e^{2t} + 1.25te^{2t} + 1.25t - 1.25$$

3. Find the following derivatives.

a) $\frac{d}{dt}(\sin t + \cos t)$

b) $\frac{d}{dt}((t+2)^{-2})$

c) $\frac{d}{dt}(5te^{8t})$

d) $\frac{d}{dt}(5\ln t)$

(ans.

a) $\frac{d}{dt}(\sin t + \cos t) = \frac{d}{dt}(\sin t) + \frac{d}{dt}(\cos t) = \cos t - \sin t$

b) $\frac{d}{dt}((t+2)^{-2}) = -2(t+2)^{-3}$

c) $\frac{d}{dt}(5te^{8t}) = 5e^{8t} + 40te^{8t}$

d) $\frac{d}{dt}(5\ln t) = \frac{5}{t}$

4. Find the following integrals

a) $\int 6t^2 dt$

b) $\int 14e^{7t} dt$

c) $\int \sin(0.5t) dt$

d) $\int \frac{5}{x} dx$

(ans.

a) $\int 6t^2 dt = 6\left(\frac{t^3}{3}\right) = 2t^3 + C$

b) $\int 14e^{7t} dt = 14\left(\frac{e^{7t}}{7}\right) + C = 2e^{7t} + C$

c) $\int \sin(0.5t) dt = \frac{-\cos(0.5t)}{0.5} + C = -2\cos(0.5t) + C$

d) $\int \frac{5}{x} dx = 5\ln(x) + C$

5. Find the following derivative.

$$\frac{d}{dt}(5te^{4t} + (t+4)^{-3})$$

6. Find the following derivatives.

a) $\frac{d}{dx}\left(\frac{1}{x+1}\right)$

b) $\frac{d}{dt}(e^{-t}\sin(2t-4))$

(ans.

a) $\frac{d}{dx}\left(\frac{1}{x+1}\right) = \frac{-1}{(x+1)^2}$

b) $\frac{d}{dt}(e^{-t}\sin(2t-4)) = -e^{-t}\sin(2t-4) + 2e^{-t}\cos(2t-4)$

7. Solve the following integrals.

a) $\int e^{2t}dt$

b) $\int(\sin\theta + \cos 3\theta)d\theta$

(ans.

a) $\int e^{2t}dt = 0.5e^{2t} + C$

b) $\int(\sin\theta + \cos 3\theta)d\theta = -\cos\theta + \frac{1}{3}\sin 3\theta + C$

8. Solve the following differential equation.

$$x'' + 5x' + 3x = 3 \qquad x(0) = 1$$
$$x'(0) = 1$$

(ans. $x'' + 5x' + 3x = 3 \qquad x(0) = 1 \qquad x'(0) = 1$

Homogeneous:

$$A^2 + 5A + 4 = 0$$

$$A = \frac{-5 \pm \sqrt{25-16}}{2} = \frac{-5 \pm 3}{2} = -4, -1$$

$$x_h = C_1 e^{-4t} + C_2 e^{-t}$$

Particular:

$$x_p = A \qquad x'_p = 0 \qquad x''_p = 0$$

$$0 + 5(0) + 3A = 3 \qquad A = 1$$

$$x_p = 1$$

Initial values:

$$x = x_h + x_p = C_1 e^{-4t} + C_2 e^{-t} + 1$$

$$1 = C_1(1) + C_2(1) + 1 \qquad C_1 = -C_2$$

$$x' = -4C_1 e^{-4t} - C_2 e^{-t}$$

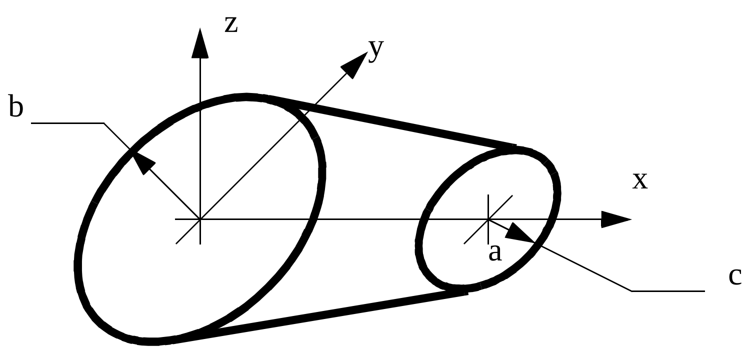
$$1 = -4C_1(1) - C_2(1)$$

$$4(-C_2) + C_2 = -1 \qquad C_2 = \frac{1}{3}$$

$$C_1 = -\frac{1}{3}$$

$$x = -\frac{1}{3}e^{-4t} + \frac{1}{3}e^{-t} + 1$$

9. Set up an integral and solve it to find the volume inside the shape below. The shape is basically a cone with the top cut off.



(ans.

$$V = \int dV = \int_0^a A dx$$

$$r = b + \left(\frac{c-b}{a}\right)x$$

$$A = \pi r^2 = \pi\left(b + \left(\frac{c-b}{a}\right)x\right)^2 = \pi b^2 + 2\pi\left(\frac{c-b}{a}\right)x + \pi\left(\frac{c-b}{a}\right)^2 x^2$$

$$V = \int_0^a \left(\pi b^2 + 2\pi\left(\frac{c-b}{a}\right)x + \pi\left(\frac{c-b}{a}\right)^2 x^2\right)dx$$

$$V = \pi b^2 x + \pi\left(\frac{c-b}{a}\right)x^2 + \frac{\pi}{3}\left(\frac{c-b}{a}\right)^2 x^3 \Big|_0^a$$

$$V = \pi b^2 a + \pi\left(\frac{c-b}{a}\right)a^2 + \frac{\pi}{3}\left(\frac{c-b}{a}\right)^2 a^3$$

$$V = \pi b^2 a + \pi(c-b)a + \frac{\pi}{3}(c-b)^2 a^2$$

10. Solve the first order non-homogeneous differential equation below. Assume the system starts at rest.

$$2x' + 4x = 5\sin 4t$$

11. Solve the second order non-homogeneous differential equation below.

$$2x'' + 4x' + 2x = 5 \qquad \text{where,} \quad x(0) = 2$$

$$x'(0) = 0$$

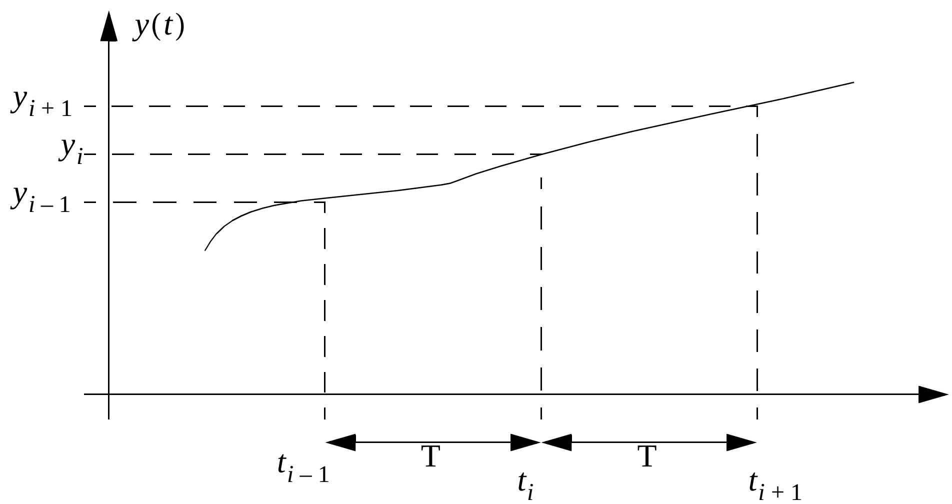
34.7 NUMERICAL METHODS

• These techniques approximate system responses without doing integrations, etc.

34.7.1 Approximation of Integrals and Derivatives from Sampled Data

• This form of integration is done numerically - this means by doing repeated calculations to solve the equation. Numerical techniques are not as elegant as solving differential equations, and will result in small errors. But these techniques make it possible to solve complex problems much faster.

• This method uses forward/backward differences to estimate derivatives or integrals from measured data.



$$\int_{t_{i-1}}^{t_i} y(t_i) \approx \left(\frac{y_i + y_{i-1}}{2}\right)(t_i - t_{i-1}) = \frac{T}{2}(y_i + y_{i-1})$$

$$\frac{d}{dt}y(t_i) \approx \left(\frac{y_i - y_{i-1}}{t_i - t_{i-1}}\right) = \left(\frac{y_{i+1} - y_i}{t_{i+1} - t_i}\right) = \frac{1}{T}(y_i - y_{i-1}) = \frac{1}{T}(y_{i+1} - y_i)$$

$$\left(\frac{d}{dt}\right)^2 y(t_i) \approx \frac{\frac{1}{T}(y_{i+1} - y_i) - \frac{1}{T}(y_i - y_{i-1})}{T} = \frac{-2y_i + y_{i-1} + y_{i+1}}{T^2}$$

34.7.2 Euler First-order Integration

• We can also estimate the change resulting from a derivative using Euler's equation for a first-order difference equation.

$$y(t+h) \approx y(t) + h\frac{d}{dt}y(t)$$

34.7.3 Taylor Series Integration

• Recall the basic Taylor series,

$$x(t+h) = x(t) + h\left(\frac{d}{dt}\right)x(t) + \frac{1}{2!}h^2\left(\frac{d}{dt}\right)^2 x(t) + \frac{1}{3!}h^3\left(\frac{d}{dt}\right)^3 x(t) + \frac{1}{4!}h^4\left(\frac{d}{dt}\right)^4 x(t) + \dots$$

• When h=0 this is called a MacLaurin series.

• We can integrate a function by,

$$\left(\frac{d}{dt}\right)x = 1 + x^2 + t^3$$

$$\left(\frac{d}{dt}\right)x_0 = 0$$

$$x_0 = 0$$

$$h = 0.1$$

t (s)	x(t)	d/dt x(t)
0	0	0
0.1		
0.2		
0.3		
0.4		
0.5		
0.6		
0.7		
0.8		
0.9		

34.7.4 Runge-Kutta Integration

• The equations below are for calculating a fourth order Runge-Kutta integration.

$$x(t+h) = x(t) + \frac{1}{6}(F_1 + 2F_2 + 2F_3 + F_4)$$

$$F_1 = hf(t, x)$$

$$F_2 = hf\left(t + \frac{h}{2}, x + \frac{F_1}{2}\right)$$

$$F_3 = hf\left(t + \frac{h}{2}, x + \frac{F_2}{2}\right)$$

$$F_4 = hf(t + h, x + F_3)$$

where,

x = the state variables
f = the differential function
t = current point in time
h = the time step to the next integration point

34.7.5 Newton-Raphson to Find Roots

• When given an equation where an algebraic solution is not feasible, a numerical solution may be required. One simple technique uses an instantaneous slope of the function, and takes iterative steps towards a solution.

$$x_{i+1} = x_i - \frac{f(x_i)}{\left(\frac{d}{dx}f(x_i)\right)}$$

• The function f(x) is supplied by the user.

• This method can become divergent if the function has an inflection point near the root.

• The technique is also sensitive to the initial guess.

• This calculation should be repeated until the final solution is found.

34.8 LAPLACE TRANSFORMS

• The Laplace transform allows us to reverse time. And, as you recall from before the inverse of time is frequency. Because we are normally concerned with response, the Laplace transform is much more useful in system analysis.

• The basic Laplace transform equations is shown below,

$$F(s) = \int_0^\infty f(t)e^{-st}dt$$

where,

$f(t)$ = the function in terms of time t

$F(s)$ = the function in terms of the Laplace s

34.8.1 Laplace Transform Tables

• Basic Laplace Transforms for operational transformations are given below,

TIME DOMAIN	FREQUENCY DOMAIN
$Kf(t)$	$Kf(s)$
$f_1(t) + f_2(t) - f_3(t) + \ldots$	$f_1(s) + f_2(s) - f_3(s) + \ldots$
$\frac{df(t)}{dt}$	$sf(s) - f(0^-)$
$\frac{d^2 f(t)}{dt^2}$	$s^2 f(s) - sf(0^-) - \frac{df(0^-)}{dt}$
$\frac{d^n f(t)}{dt^n}$	$s^n f(s) - s^{n-1} f(0^-) - s^{n-2} \frac{df(0^-)}{dt} - \ldots - \frac{d^n f(0^-)}{dt^n}$
$\int_0^t f(t) dt$	$\frac{f(s)}{s}$
$f(t-a)u(t-a), a > 0$	$e^{-as} f(s)$
$e^{-at} f(t)$	$f(s-a)$
$f(at), a > 0$	$\frac{1}{a} f\left(\frac{s}{a}\right)$
$tf(t)$	$\frac{-df(s)}{ds}$
$t^n f(t)$	$(-1)^n \frac{d^n f(s)}{ds^n}$
$\frac{f(t)}{t}$	$\int_s^\infty f(u) du$

• A set of useful functional Laplace transforms are given below,

TIME DOMAIN	FREQUENCY DOMAIN
A	$\frac{A}{s}$
t	$\frac{1}{s^2}$
e^{-at}	$\frac{1}{s+a}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
te^{-at}	$\frac{1}{(s+a)^2}$
$e^{-at}\sin(\omega t)$	$\frac{\omega}{(s+a)^2+\omega^2}$
$e^{-at}\cos(\omega t)$	$\frac{s+a}{(s+a)^2+\omega^2}$
Ae^{-at}	$\frac{A}{s-a}$
Ate^{-at}	$\frac{A}{(s-a)^2}$
$2\lvert A\rvert e^{-\alpha t}\cos(\beta t+\theta)$	$\frac{A}{s+\alpha-\beta j}+\frac{A^{\text{complex conjugate}}}{s+\alpha+\beta j}$
$2t\lvert A\rvert e^{-\alpha t}\cos(\beta t+\theta)$	$\frac{A}{(s+\alpha-\beta j)^2}+\frac{A^{\text{complex conjugate}}}{(s+\alpha+\beta j)^2}$

• Laplace transforms can be used to solve differential equations.

34.9 z-TRANSFORMS

• For a discrete-time signal $x[n]$, the two-sided z-transform is defined by $X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$. The one-sided z-transform is defined by $X(z) = \sum_{n=0}^{\infty} x[n]z^{-n}$. In both cases, the z-transform is a polynomial in the complex variable z.

• The inverse z-transform is obtained by contour integration in the complex plane $x[n] = \frac{1}{j2\pi}\oint X(z)z^{n-1}dz$. This is usually avoided by partial fraction inversion techniques, similar to the Laplace transform.

• Along with a z-transform we associate its region of convergence (or ROC). These are the values of z for which $X(z)$ is bounded (i.e., of finite magnitude).

• Some common z-transforms are shown below.

Table 1: Common z-transforms

Signal $x[n]$	z-Transform $X(z)$	ROC
$\delta[n]$	1	All z
$u[n]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
$nu[n]$	$\frac{z^{-1}}{(1-z^{-1})^2}$	$\lvert z\rvert > 1$
$n^2u[n]$	$\frac{z^{-1}(1+z^{-1})}{(1-z^{-1})^3}$	$\lvert z\rvert > 1$
$a^nu[n]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
$na^nu[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
$(-a^n)u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
$(-na^n)u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
$\cos(\omega_0 n)u[n]$	$\frac{1-z^{-1}\cos\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$	$\lvert z\rvert > 1$
$\sin(\omega_0 n)u[n]$	$\frac{z^{-1}\sin\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$	$\lvert z\rvert > 1$
$a^n\cos(\omega_0 n)u[n]$	$\frac{1-az^{-1}\cos\omega_0}{1-2az^{-1}\cos\omega_0+a^2z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$

Table 1: Common z-transforms

Signal $x[n]$	z-Transform $X(z)$	ROC
$a^n \sin(\omega_0 n) u[n]$	$\dfrac{az^{-1}\sin\omega_0}{1 - 2az^{-1}\cos\omega_0 + a^2 z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$
$\dfrac{n!}{k!(n-k)!} u[n]$	$\dfrac{z^{-k}}{(1-z^{-1})^{k+1}}$	$\lvert z\rvert > 1$

• The z-transform also has various properties that are useful. The table below lists properties for the two-sided z-transform. The one-sided z-transform properties can be derived from the ones below by considering the signal $x[n]u[n]$ instead of simply $x[n]$.

Table 2: Two-sided z-Transform Properties

Property	Time Domain	z-Domain	ROC
Notation	$x[n]$ $x_1[n]$ $x_2[n]$	$X(z)$ $X_1(z)$ $X_2(z)$	$r_2 < \lvert z\rvert < r_1$ ROC_1 ROC_2
Linearity	$\alpha x_1[n] + \beta x_2[n]$	$\alpha X_1(z) + \beta X_2(z)$	At least the intersection of ROC_1 and ROC_2
Time Shifting	$x[n-k]$	$z^{-k} X(z)$	That of $X(z)$, except $z = 0$ if $k > 0$ and $z = \infty$ if $k < 0$
z-Domain Scaling	$a^n x[n]$	$X(a^{-1}z)$	$\lvert a\rvert r_2 < \lvert z\rvert < \lvert a\rvert r_1$
Time Reversal	$x[-n]$	$X(z^{-1})$	$\dfrac{1}{r_1} < \lvert z\rvert < \dfrac{1}{r_2}$
z-Domain Differentiation	$nx[n]$	$-z\dfrac{dX(z)}{dz}$	$r_2 < \lvert z\rvert < r_1$

Table 2: Two-sided z-Transform Properties

Property	Time Domain	z-Domain	ROC
Convolution	$x_1[n]*x_2[n]$	$X_1(z)X_2(z)$	At least the intersection of ROC_1 and ROC_2
Multiplication	$x_1[n]x_2[n]$	$\frac{1}{j2\pi}\oint X_1(v)X_2\left(\frac{z}{v}\right)v^{-1}dv$	At least $r_{1l}r_{2l} < \|z\| < r_{1u}r_{2u}$
Initial value theorem	$x[n]$ causal	$x[0] = \lim_{z \to \infty} X(z)$	

34.10 FOURIER SERIES

• These series describe functions by their frequency spectrum content. For example a square wave can be approximated with a sum of a series of sine waves with varying magnitudes.

• The basic definition of the Fourier series is given below.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)\right]$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx \qquad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

34.11 TOPICS NOT COVERED (YET)

• To ensure that the omissions are obvious, I provide a list of topics not covered below. Some of these may be added later if their need becomes obvious.

• Frequency domain - Fourier, Bessel

34.12 REFERENCES/BIBLIOGRAPHY

Spiegel, M. R., Mathematical Handbook of Formulas and Tables, Schaum's Outline Series, McGraw-Hill Book Company, 1968.